CLASSICAL r-MATRIX LIKE APPROACH TO FROBENIUS MANIFOLDS, WDVV EQUATIONS AND FLAT METRICS

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Abstract. A general scheme for construction of flat pencils of contravariant metrics and Frobenius manifolds as well as related solutions to WDVV associativity equations is formulated. The advantage is taken from the Rota-Baxter identity and some relation being counterpart of the modified Yang-Baxter identity from the classical r-matrix formalism. The scheme for the construction of Frobenius manifolds is illustrated on the algebras of formal Laurent series and meromorphic functions on Riemann sphere.

Contents

1. Introduction

Classical r-matrix theory [30] can be very useful in the construction of almost all known classes of integrable field systems (see [31, 6] and references therein). Frobenius manifolds are intrinsically connected with the bi-Hamiltonian systems of hydrodynamic type, for which the above formalism can be naturally applied (see for instance [26, 34]). Therefore, the primary objective of this article is a formulation of a scheme analogous to the classical r-matrix theory, which could be applied for the construction of Frobenius manifolds.

The main idea exploited in the paper involves the use of the Rota-Baxter algebras for the construction of Frobenius algebras. Consider the new multiplication

$$a \circ b := \ell(a)b + a\ell(b)$$

defined in some commutative associative algebra. If the endomorphism ℓ satisfies the Rota-Baxter identity [5, 29, 20]:

$$\ell(\ell(a)b) + \ell(a\ell(b)) - \ell(a)\ell(b) = \kappa\, ab,$$

of weight κ, then the new multiplication is associative. If the algebra can be equipped with some trace form, then the following metric is automatically invariant:

$$\eta(a,b) := \mathrm{Tr}(a \circ b) \quad \Rightarrow \quad \eta(a\circ b, c) = \eta(a, b\circ c)\,.$$

Hence, if the new multiplication is unital we can generate in this way, in principal, nontrivial Frobenius algebras. We show in the article how this idea can be extended to Frobenius algebras, appearing in the cotangent bundles, of certain Frobenius manifolds.

The theory of Frobenius manifolds [13, 14] is a coordinate-free formulation of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) associativity equations appearing in the context of the 2-dimensional topological quantum field theories (TFT) [35, 10]. Frobenius manifolds appear not only in theoretical physics but also in seemingly unrelated areas of contemporary mathematics such as enumerative geometry and quantum cohomology [27], singularity theory [21] and integrable systems [14, 19], see also the expository articles [22, 4].

In fact the structure of Frobenius manifold is equivalent to a pencil of flat metrics satisfying some homogeneity conditions [14, 15]. These pencils generate hydrodynamic (Dubrovin-Novikov) Poisson tensors yielding bi-Hamiltonian structure for the so-called principal hierarchy that can be associated to any Frobenius manifold. Such integrable hierarchies together with their bi-Hamiltonian structures can be efficiently constructed using the classical r-matrix formalism (see for instance [34]).

More standard approach to the construction of Frobenius manifolds relies on the Landau-Ginzburg models and Saito's theory [14]. It is interesting that often in these formalisms the superpotentials can be identified with Lax functions of the related Lax hierarchies. At the general level, it does not seem to be so apparent and the further research on the mutual connections is justified. The above point of view for construction of Frobenius manifolds is presented for instance in the following articles [25, 2, 3, 32, 33, 8], which are directly connected with integrable hierarchies of hydrodynamic type so-called

Whitham hierarchies. Alternative approach based on the so-called isotropic deformations is presented in [23, 24]. For some recent works concerning classification of semisimple Frobenius manifolds see [11, 17, 12].

A new interesting class of Frobenius manifolds, which are infinite-dimensional, was introduced in the recent articles [9] and [28]. These Frobenius manifolds are associated with (2+1)-dimensional integrable hydrodynamic 2d Toda and KP equations, respectively. Both works rely on the bi-Hamiltonian structures of related infinite-field hydrodynamic chains. Nevertheless, both approaches are not equivalent as they exploit in the construction of Frobenius structures significantly different mathematical methods. The approach from [9] is further extended in [36] to infinite-dimensional Frobenius manifolds associated with two-component dispersionless BKP hierarchy. In principle, the approach presented in this article can be used in the above cases of infinite-dimensional Frobenius manifolds.

In Section 2 we present all the facts about Frobenius manifolds and related structures that will be indispensable in the rest of the paper. In particular Proposition 2.4 will allow for the straightforward construction of solutions to the WDVV equations. In Section 3 we establish scheme for the construction of Frobenius algebras, which is based on the Rota-Baxter identity (3.2). In Section 4 we present scheme for the construction of flat metrics, which is based on the relation (4.4) being counterpart of the modified Yang-Baxter identity from the classical r-matrix formalism. In Section 5 the scheme for the construction of Frobenius manifolds is developed, which is based on the results from two preceding sections. Here, we prove the main Theorem 5.4 of this article. In the last Section 6 we illustrate our scheme of construction of Frobenius manifolds applying it to the algebras of formal Laurent series and meromorphic functions on Riemann sphere. As a result, we generate infinite- and finite-dimensional Frobenius manifolds, respectively.

We believe that the approach presented in this article will contribute to a better understanding of the relation, on the constructive level, between Frobenius manifolds and integrable hydrodynamic systems. This follows from the fact that the scheme for the construction of Frobenius manifolds is formulated purely in the cotangent bundle, which is more natural in the context of the related hydrodynamic bi-Hamiltonian structures. Moreover, we hope that this work will contribute to the further classification of Frobenius manifolds, including particularly these which are infinite-dimensional.

2. Theory of Frobenius manifolds

2.1. **Frobenius manifolds.** In this section we present all the necessary facts about Frobenius manifolds and related structures to make the article self-contained. For the convention used see Appendix A.

Definition 2.1 ([14]). *A Frobenius manifold is an n-dimensional smooth manifold*[1] *equipped with a (pseudo-Riemannian) covariant metric $\eta \in \Gamma(S^2T^*M)$ and structure of a Frobenius algebra on the tangent bundle TM. The last statement means that there exists an unital*

[1] All structures will be considered over field of real or complex numbers, that is $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

commutative associative multiplication, given by the $\mathcal{C}^\infty(M)$-bilinear map

$$* : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \tag{2.1}$$

compatible with the metric:

$$\eta(X, Y * Z) = \eta(X * Y, Z) \qquad X, Y, Z \in \mathfrak{X}(M). \tag{2.2}$$

Let ∇ be the Levi-Civita connection of η. The following conditions are also required:

(1) The metric η must be flat.

(2) The tensor field ∇c must be symmetric in all its four arguments, where $c(X, Y, Z) := \eta(X * Y, Z)$.

(3) The unit vector field e must be flat, that is $\nabla e = 0$.

(4) There must exists an Euler vector field E (i.e. $\nabla\nabla E = 0$) such that

$$\mathrm{Lie}_E * = * \qquad \textit{and} \qquad \mathrm{Lie}_E\, \eta = (2 - d)\, \eta, \tag{2.3}$$

where d is some number (weight).

The above structure without point (4) will be referred to as a pre-Frobenius manifold.

According to Theorem 2.15 in [21] the following conditions on any Frobenius manifold are equivalent:

i) The tensor ∇c is symmetric in all four arguments.

ii) The tensor $\nabla *$ is symmetric in all three arguments.

iii) The multiplication (2.1) satisfies the F-manifold condition,

$$\mathrm{Lie}_{X*Y}(*) = X * \mathrm{Lie}_Y(*) + Y * \mathrm{Lie}_X(*), \tag{2.4}$$

and the counity 1-form $\varepsilon := e^\flat$ is closed.

Moreover, from Lemma 2.16 in [21] we know that the vector field e is flat if and only if

$$\mathrm{Lie}_e\, \eta = 0 \tag{2.5}$$

and $d\varepsilon = 0$. Hence, in Definition 2.1 instead of the third condition we can require (2.5) to hold. Direct consequence of (2.4) is that $\mathrm{Lie}_e(*) = 0$ and $\mathrm{Lie}_E\, e = -e$.

Substituting e for Z in (2.2) one observes that the counity is actually a trace form, $\varepsilon : \mathfrak{X}(M) \to \mathcal{C}^\infty(M)$, such that $\eta(X, Y) = \varepsilon(X * Y)$.

2.2. **WDVV associativity equations.** Let $t^1, \dots, t^n$ be (local) flat coordinates for the metric η such that $e = \partial_{t^1}$. Then, the second condition in Definition 2.1 implies local existence of the (smooth) function $\mathcal{F} = \mathcal{F}(t)$, the so-called prepotential, such that

$$c_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial t^i \partial t^j \partial t^k} \qquad \text{and} \qquad \eta_{ij} = \frac{\partial^3 \mathcal{F}}{\partial t^1 \partial t^i \partial t^j}. \tag{2.6}$$

The structure constants for the multiplication (2.1), such that $(X * Y)^i = c^i_{jk} X^j Y^k$, are given by $c^i_{jk} = \eta^{il} c_{ljk}$.[2] Then, the so-called WDVV equations are the associativity equations on the prepotential $\mathcal{F}$:

$$\frac{\partial^3 \mathcal{F}}{\partial t^i \partial t^j \partial t^r} \eta^{rs} \frac{\partial^3 \mathcal{F}}{\partial t^s \partial t^k \partial t^l} = \frac{\partial^3 \mathcal{F}}{\partial t^l \partial t^j \partial t^r} \eta^{rs} \frac{\partial^3 \mathcal{F}}{\partial t^s \partial t^k \partial t^i}.$$

[2] In this section the Einstein summation convention is used.

The Euler vector field E can be normalised so that $E^i = (1-q_i)t^i + r_i$, where q_i, r_i are some numbers. Furthermore, the quasi-homogeneity condition (2.3) on $\mathcal{F}$ takes the form

$$\mathrm{Lie}_E\,\mathcal{F} \equiv E^i\frac{\partial \mathcal{F}}{\partial t^i} = (3-d)\mathcal{F} + \textit{quad. pol.} \tag{2.7}$$

The above equality holds modulo quadratic polynomials in the flat coordinates.

2.3. **Intersection form.** On a Frobenius manifold the metric η induces structure of a Frobenius algebra in the cotangent bundle T^*M with the multiplication given by

$$\alpha\circ\beta := \left(\alpha^\sharp * \beta^\sharp\right)^\flat \qquad \alpha,\beta\in\Lambda^1(M).$$

Its unity is the counity 1-form $\varepsilon = e^\flat$ and the unity vector field e is a trace form such that $\eta^*(\alpha,\beta) = e(\alpha\circ\beta)$. The quasi-homogeneity relations (2.3) can be rewritten in the form:

$$\mathrm{Lie}_E \circ = (d-1)\circ, \qquad \mathrm{Lie}_E\,\eta^* = (d-2)\,\eta^*. \tag{2.8}$$

Besides, on a Frobenius manifold there exists a second contravariant metric $g^*\in\Gamma(S^2TM)$, the so-called intersection form [14, 15], defined by

$$g^*(\alpha,\beta) := \langle \alpha\circ\beta, E\rangle \qquad \alpha,\beta\in\Lambda^1(M), \tag{2.9}$$

where E is the Euler vector field. In fact, this metric is also flat and its inverse g together with η are compatible, that is the (covariant) pencil defined by $g^z := \eta + z\,g$ is flat for all values of z.

Remark 2.2. The existence of the intersection form g is the main source of the connection of Frobenius manifolds with integrable systems of hydrodynamic type. This is because the flat metrics η and g generate compatible Poisson brackets of hydrodynamic type, see Appendix B.

2.4. **Deformed flat connection.** On a Frobenius manifold one can define affine connection in the form

$$\widetilde{\nabla}_X Y := \nabla_X Y + z\,X * Y \qquad X,Y\in\mathfrak{X}(M), \tag{2.10}$$

where $z\in\mathbb{C}^*$ is a deformation parameter. This connection is torsionless (symmetric) and its curvature tensor vanish identically in z. The symmetry of (2.10) is equivalent to the commutativity of the multiplication (2.1) and its flatness is equivalent to the associativity of (2.1) as well as symmetry of the tensor ∇c with respect to all its arguments.

Proposition 2.3. *The action of the deformed connection* (2.10) *on* 1*-forms is given by*

$$\widetilde{\nabla}_{\alpha^\sharp}\gamma = \nabla_{\alpha^\sharp}\gamma - z\,\alpha\circ\gamma, \tag{2.11}$$

where $\alpha,\gamma\in\Lambda^1(M)$.

Proof. By the invariance (2.2) one finds that

$$\langle\gamma, \alpha^\sharp * X\rangle = \eta(\gamma^\sharp, \alpha^\sharp * X) = \eta(\alpha^\sharp * \gamma^\sharp, X) = \langle \alpha\circ\gamma, X\rangle.$$

Hence, using the properties of affine connection we have

$$\begin{aligned}\langle\widetilde{\nabla}_{\alpha^\sharp}\gamma, X\rangle &= \mathrm{D}_{\alpha^\sharp}\langle\gamma, X\rangle - \langle\gamma, \widetilde{\nabla}_{\alpha^\sharp}X\rangle = \langle\nabla_{\alpha^\sharp}\gamma, X\rangle - z\,\langle\gamma, \alpha^\sharp * X\rangle\\ &= \langle\nabla_{\alpha^\sharp}\gamma, X\rangle - z\,\langle\alpha\circ\gamma, X\rangle,\end{aligned}$$

which gives the formula (2.10). □

Flatness of the deformed connection $\widetilde{\nabla}$ entails (local) existence of its flat coordinates $\mathcal{H}^k(z) \equiv \mathcal{H}^k(t,z)$ such that

$$\widetilde{\nabla}_j d\mathcal{H}^k(z) = 0 \qquad \Longleftrightarrow \qquad \frac{\partial^2\mathcal{H}^k(z)}{\partial t^i\partial t^j} = z\,c^l_{ij}\,\frac{\partial\mathcal{H}^k(z)}{\partial t^l},$$

where $\nabla_j \equiv \nabla_{\frac{\partial}{\partial t^j}}$. One can expand $\mathcal{H}^k(z)$ into the formal power series

$$\mathcal{H}^k(z) := \sum_{n=0}^{\infty}\mathcal{H}^k_{(n)}(t)z^n.$$

Then, the coefficient functions $\mathcal{H}^k_{(n)}(t)$ can be determined recursively from

$$\frac{\partial^2\mathcal{H}^k_{(n)}}{\partial t^i\partial t^j} = c^l_{ij}\,\frac{\partial\mathcal{H}^k_{(n-1)}}{\partial t^l} \qquad n>0.$$

This recurrence formula has the following coordinate-free form:

$$\nabla_{\alpha^\sharp}d\mathcal{H}^k_{(n)} = \alpha\circ d\mathcal{H}^k_{(n-1)} \qquad n>0, \tag{2.12}$$

valid for arbitrary $\alpha\in\Lambda^1(M)$.

Proposition 2.4. *Assuming normalization $\mathcal{H}^k_{(0)} = t^k$, if $d\neq 3$ the prepotential $\mathcal{F}$ can be determined from $\mathcal{H}^k_{(1)}$ using the formula*

$$\mathcal{F} = \frac{1}{3-d}\sum_{i,j}\eta_{ij}E^i\mathcal{H}^j_{(1)} + quad.\ pol., \tag{2.13}$$

where the equality holds modulo quadratic polynomials in the flat coordinates.

Proof. Using the normalization and (2.6) we have

$$\frac{\partial^2\mathcal{H}^k_{(1)}}{\partial t^i\partial t^j} = c^l_{ij}\frac{\partial\mathcal{H}^k_{(0)}}{\partial t^l} = c^k_{ij} = \eta^{kl}c_{ijl} = \eta^{kl}\frac{\partial^3\mathcal{F}}{\partial t^i\partial t^j\partial t^l}.$$

Integrating twice, one finds

$$\mathcal{H}^k_{(1)} = \eta^{kl}\frac{\partial\mathcal{F}}{\partial t^l} + lin.\ pol. \quad \Rightarrow \quad \frac{\partial\mathcal{F}}{\partial t^i} = \eta_{ik}\mathcal{H}^k_{(1)} + lin.\ pol.\ ,$$

where the equalities hold modulo linear functions. Next, substituting this to (2.7) one obtains the desired formula (2.13). □

3. Construction of Frobenius algebras

3.1. **Rota-Baxter identity.** Recall that a Frobenius algebra is an associative commutative unital algebra $\mathcal{A}$ endowed with a nondegenerate invariant symmetric bilinear scalar form.

Proposition 3.1. *Let $\mathcal{A}$ be an associative (non-necessarily commutative) algebra.*

i) Define new multiplication on $\mathcal{A}$:

$$a \circ_\ell b := \ell(a)b + a\ell(b) \qquad a, b \in \mathcal{A}, \tag{3.1}$$

generated by some linear map $\ell : \mathcal{A} \to \mathcal{A}$. A sufficient condition for the multiplication (3.1) to be associative is the identity

$$\ell(a \circ_\ell b) - \ell(a)\ell(b) = \kappa\, ab, \tag{3.2}$$

which must hold for all $a, b \in \mathcal{A}$ and some $\kappa \in \mathrm{Z}(\mathcal{A})$ (center of $\mathcal{A}$).

ii) Moreover, the linear map $\widetilde{\ell}(\cdot) = \ell(\delta\,\cdot)$, where ℓ is composed with some $\delta \in \mathrm{Z}(\mathcal{A})$, satisfies (3.2), with κ replaced by $\widetilde{\kappa} = \kappa\,\delta^2$, iff ℓ satisfies (3.2).

Proof. From the formula (3.2),

$$\begin{aligned}(a \circ_\ell b) \circ_\ell c - a \circ_\ell (b \circ_\ell c) &= \ell(a \circ_\ell b)\, c + (a \circ_\ell b)\, \ell(c) - \ell(a)(b \circ_\ell c) - a\, \ell(b \circ_\ell c) = \\ &= [\ell(a \circ_\ell b) - \ell(a)\ell(b)]\, c - a[\ell(b \circ_\ell c) - \ell(b)\ell(c)] = \kappa abc - a\kappa bc.\end{aligned}$$

Thus, the first assertion follows immediately if κ commutes with all elements from $\mathcal{A}$. For $\widetilde{\ell}$ the right-hand side of (3.2) takes the form

$$\widetilde{\ell}(a \circ_{\widetilde{\ell}} b) - \widetilde{\ell}(a)\widetilde{\ell}(b) = \ell((\delta a) \circ_\ell (\delta b)) - \ell(\delta a)\ell(\delta b) = \kappa\delta^2 ab.$$

Hence, the second assertion follows. □

Remark 3.2. The formula (3.2) is known as the Rota-Baxter identity [5, 29]. In most cases, when $\mathcal{A}$ is unital, κ is a scalar weight. Associative algebras equipped with an operator satisfying the identity (3.2) are called Rota-Baxter algebras, for information on the subject see [20] and references therein.

3.2. **Invariant scalar product.** Let $\mathcal{A}$ be a commutative associative unital algebra with a trace form given by a linear map $\mathrm{Tr} : \mathcal{A} \to \mathbb{K}$ such that the pairing

$$(\cdot,\cdot)_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathbb{K} \qquad (a, b)_{\mathcal{A}} := \mathrm{Tr}(ab),$$

is nondegenerate. We will call such trace nondegenerate.

On the other hand, for an unital associative commutative algebra $\mathcal{A}$ equipped with a nondegenerate invariant pairing the trace form can be always defined as $\mathrm{Tr}\, a := (a, 1)_{\mathcal{A}}$. This pairing is naturally invariant, hence such $\mathcal{A}$ is a Frobenius algebra. However, our aim is the construction of a 'more complex' Frobenius structure on $\mathcal{A}$ with scalar product invariant with respect to the commutative multiplication (3.1).

Let $\circ_\ell$ defined by (3.1) be a second commutative associative multiplication on $\mathcal{A}$. Then, we can define bilinear form (metric):

$$\eta(a, b) := \mathrm{Tr}(a \circ_\ell b) \qquad a, b \in \mathcal{A}, \tag{3.3}$$

naturally invariant with respect to the multiplication $\circ_\ell$, that is

$$\eta(a \circ_\ell b, c) = \mathrm{Tr}(a \circ_\ell b \circ_\ell c) = \eta(a, b \circ_\ell c)\,.$$

If the new multiplication (3.1) is unital and such that (3.3) is nondegenerate, then the multiplication (3.1) together with the metric (3.3) define structure of a Frobenius algebra on $\mathcal{A}$.

The relation from the following proposition will be needed later.

Proposition 3.3. *For the algebra $\mathcal{A}$ endowed with a nondegenerate inner product, the Rota-Baxter identity* (3.2) *is equivalent to the following 'dual' relation:*

$$\ell^*(\ell^*(a)b) - \ell^*(a\,\ell(b)) + \ell^*(a)\ell(b) = \kappa\, ab, \tag{3.4}$$

where ℓ^* is the adjoint of ℓ such that $\mathrm{Tr}(\ell^*(a)b) := \mathrm{Tr}(a\ell(b))$.

Proof. Define functionals in the form:

$$\begin{aligned}
K_1[a,b] &:= \ell(\ell(a)b) + \ell(a\,\ell(b)) - \ell(a)\ell(b) - \kappa\ ab,\\
K_2[a,b] &:= \ell^*(\ell^*(a)b) - \ell^*(a\,\ell(b)) + \ell^*(a)\ell(b) - \kappa\, ab,
\end{aligned}$$

which vanishing is equivalent to the identities (3.2) and (3.4). The lemma follows from the equality

$$\mathrm{Tr}(K_1[a,b]\,c) = \mathrm{Tr}(aK_2[c,b])$$

and the fact that the inner product defined by the trace form is assumed to be nondegenerate. □

3.3. **Special case.** There is a class of simple solutions to the Rota-Baxter identity (3.2) that will be of interest to us. Assume that the algebra $\mathcal{A}$ can be decomposed into a (vector) direct sum of subalgebras preserving the multiplication, that is

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \qquad \mathcal{A}_\pm \mathcal{A}_\pm \subset \mathcal{A}_\pm \qquad \mathcal{A}_+ \cap \mathcal{A}_- = \emptyset.$$

Denoting the projections onto this subalgebras by $P_\pm$, we define $\ell : \mathcal{A} \to \mathcal{A}$ by

$$\ell = \frac{1}{2}(P_+ - P_-)\,. \tag{3.5}$$

Proposition 3.4. *The linear map* (3.5) *satisfies the identity* (3.2) *for $\kappa = \frac{1}{4}$.*

Proof. Set $a_\pm \equiv P_\pm(a)$ for $a \in \mathcal{A}$, thus $a = a_+ + a_-$. Then,

$$\begin{aligned}
\ell(a \circ_\ell b) &= \ell(a_+b_+ - a_-b_-) = \frac{1}{2}(a_+b_+ + a_-b_-)\,,\\
\ell(a)\ell(b) &= \frac{1}{4}(a_+b_+ - a_+b_- - a_-b_+ + a_-b_-)\,.
\end{aligned}$$

Now, the result follows from a simple verification. □

Remark 3.5. The above special solution to the Rota-Baxter identity is a counterpart of similar construction for the modified Yang-Baxter equation, see Appendix C.

4. Construction of flat metrics

Let $\mathcal{A}$ be a commutative associative unital algebra equipped with a trace form $\operatorname{Tr} : \mathcal{A} \to \mathbb{K}$, such that the symmetric product $(a, b)_{\mathcal{A}} := \operatorname{Tr}(ab)$ is nondegenerate. Furthermore, let be given some derivation $\partial \in \operatorname{Der}\mathcal{A}$ invariant with respect to the trace form, that is

$$\operatorname{Tr}(a'b) = -\operatorname{Tr}(ab')\,, \tag{4.1}$$

where $a' \equiv \partial a$.

Let $\mathcal{A}_M \subset \mathcal{A}$ constitute a subspace (submanifold) of $\mathcal{A}$, such that the endomorphism ∂ is transversal to $\mathcal{A}_M$. This requirement means that all vector fields (derivations) on $\mathcal{A}_M$, $\mathfrak{X}(\mathcal{A}_M)$, must commute with ∂. Obviously, $\mathfrak{X}(\mathcal{A}_M)$ can be identified with an appropriate subset of $\operatorname{Der}\mathcal{A}$. Notice that $\mathcal{A}_M$ does not have to be a vector subspace.

We will identify $\mathcal{A}_M$ with an underlying manifold M. The tangent space $T_\lambda\mathcal{A}_M$ at some point $\lambda \in \mathcal{A}_M$ is a subset of $\mathcal{A}$. Using the nondegenerate symmetric product we can identify another subset as the cotangent space $T^*_\lambda\mathcal{A}_M$, so that the duality pairing at $\lambda \in \mathcal{A}_M$ takes the form

$$\langle\ ,\ \rangle_\lambda : T^*_\lambda\mathcal{A}_M \times T_\lambda\mathcal{A}_M \to \mathbb{K} \qquad \langle\alpha, X\rangle_\lambda := (X, \alpha)_{\mathcal{A}} = \operatorname{Tr}(X\alpha). \tag{4.2}$$

4.1. **Linear metric.** For some operator $r \in \operatorname{End}\mathcal{A}$ we define at some point $\lambda \in \mathcal{A}_M$ the contravariant metric by

$$\eta^*_\lambda(\alpha, \beta) := \operatorname{Tr}(\lambda' r(\alpha)\beta + \lambda'\alpha\, r(\beta))\,, \tag{4.3}$$

where $\alpha, \beta \in T^*_\lambda\mathcal{A}_M$. This metric is linear in λ. Then, the related canonical isomorphism $\sharp|_\lambda : T^*_\lambda\mathcal{A}_M \to T_\lambda\mathcal{A}_M$, such that $\eta^*(\alpha, \beta) = \operatorname{Tr}\big(\alpha^\sharp\beta\big)$, is given by

$$\alpha^\sharp\big|_\lambda = \lambda' r(\alpha) + r^*(\lambda'\alpha),$$

where r^* is the adjoint of r, i.e. $\operatorname{Tr}(r^*(a)b) := \operatorname{Tr}(ar(b))$. For the nondegeneracy of the metric (4.3) we require the kernel of $\sharp$ to be trivial at arbitrary $\lambda \in \mathcal{A}_M$. In practice, this requirement is possible to satisfy only outside some discriminant.

The following identity on the endomorphism r turns out to be important:

$$r(r(a)b') + r(a\, r(b)') - r(a)r(b)' = \kappa\, ab' \qquad a, b \in \mathcal{A}, \tag{4.4}$$

where κ is some constant.

Theorem 4.1. *Assume that $r \in \operatorname{End}\mathcal{A}$ is invariant on $\mathcal{A}_M$, that is r commutes with directional derivatives with respect to all vector fields $\mathfrak{X}(\mathcal{A}_M)$. Then, the following statements are valid:*

(i) If r satisfies (4.4), the Levi-Civita connection of the metric (4.3) has the form

$$\nabla_{\alpha^\sharp}\gamma = \mathrm{D}_{\alpha^\sharp}\gamma - r(\alpha)\gamma' - \alpha\, r(\gamma)' \qquad \alpha, \gamma \in \Lambda^1(\mathcal{A}_M). \tag{4.5}$$

(ii) The identity (4.4) is a sufficient condition for the metric to be flat. This means that if r satisfies (4.4) the curvature tensor vanish on $\mathcal{A}_M$, that is

$$R(\alpha^\sharp, \beta^\sharp)\gamma \equiv \nabla_{\alpha^\sharp}\nabla_{\beta^\sharp}\gamma - \nabla_{\beta^\sharp}\nabla_{\alpha^\sharp}\gamma - \nabla_{[\alpha^\sharp, \beta^\sharp]}\gamma = 0,$$

where $\alpha, \beta, \gamma \in \Lambda^1(\mathcal{A}_M)$.

Proof. The tensor field (A.8) corresponding to the connection (4.5) has the form

$$\Gamma_\alpha\gamma = \mathrm{D}_{\alpha^\sharp}\gamma - \nabla_{\alpha^\sharp}\gamma = r(\alpha)\gamma' + \alpha\, r(\gamma)'.$$

We must to show that this is the Levi-Civita connection of the metric (4.3), that is the conditions (A.9) and (A.10) are satisfied.

Let $a.b := r(a)b + a\,r(b)$, so that $\eta^*(\alpha,\beta) = \mathrm{Tr}(\lambda'\alpha.\beta)$. Then, the identity (4.4) can be written in the form:

$$r(\Gamma_a b) = r(a)r(b)' + \kappa\, ab'. \tag{4.6}$$

Straightforward computation, using (4.6), leads to the following two relations:

$$a.\Gamma_b c = b.\Gamma_a c, \qquad (a.b)' = \Gamma_a b + \Gamma_b a. \tag{4.7}$$

Now, the condition (A.9) is immediate as

$$\eta^*(\alpha,\Gamma_\beta\gamma) = \mathrm{Tr}(\lambda'\alpha.\Gamma_\beta\gamma) = \mathrm{Tr}(\lambda'\beta.\Gamma_\alpha\gamma) = \eta^*(\Gamma_\alpha\gamma,\beta)\,. \tag{4.8}$$

From the requirement that ∂ and r commute with vector fields on $\mathcal{A}_M$ it follows that $\mathrm{D}_{\alpha^\sharp}\lambda' = (\alpha^\sharp)'$ and that Γ is constant on $\mathcal{A}_M$, that is $(\mathrm{D}_{\alpha^\sharp}\Gamma)(\beta,\gamma) = 0$. Hence, the directional derivative of the metric (4.3), by (4.1) and (4.7), is

$$\begin{aligned}\left(\mathrm{D}_{\alpha^\sharp}\eta^*\right)(\beta,\gamma) &= \mathrm{Tr}\big((\alpha^\sharp)'\beta.\gamma\big) = -\,\mathrm{Tr}\big(\alpha^\sharp(\beta.\gamma)'\big) = -\eta^*(\alpha,(\beta.\gamma)')\\ &= -\eta^*(\alpha,\Gamma_\beta\gamma) - \eta^*(\alpha,\Gamma_\gamma\beta)\,.\end{aligned}$$

Using (4.8) we get the second condition (A.10):

$$\left(\mathrm{D}_{\alpha^\sharp}\eta^*\right)(\beta,\gamma) = -\eta^*(\beta,\Gamma_\alpha\gamma) - \eta^*(\Gamma_\alpha\beta,\gamma)\,.$$

Hence, indeed the formula (4.5) defines the Levi-Civita connection.

Since Γ is constant on $\mathcal{A}_M$ the curvature tensor (A.12) for the metric (4.3) takes the form

$$R(\alpha^\sharp,\beta^\sharp)\gamma = \Gamma_\alpha\Gamma_\beta\gamma - \Gamma_\beta\Gamma_\alpha\gamma - \Gamma_{\Gamma_\alpha\beta}\gamma + \Gamma_{\Gamma_\beta\alpha}\gamma = 0,$$

where the last equality is a consequence of (4.6) and straightforward computation. Thus, the metric is flat. □

4.2. **General case.** We define the generalised contravariant metric, for $r \in \mathrm{End}\,\mathcal{A}$, by

$$\begin{aligned} g^*_\lambda(\alpha,\beta) &:= \mathrm{Tr}(\lambda' r(E\alpha)\beta + \lambda'\alpha\, r(E\beta))\\ &\equiv \mathrm{Tr}(E[r^*(\lambda'\alpha)\beta + \alpha\, r^*(\lambda'\beta)])\,,\end{aligned} \tag{4.9}$$

where $\lambda \in \mathcal{A}_M$ and $\alpha,\beta \in T^*_\lambda\mathcal{A}_M$. We assume that $E : \mathcal{A}_M \to \mathcal{A}_M$ is a differentiable function $E = E(\lambda)$ of λ, such that $\mathrm{D}_X E = \frac{\partial E}{\partial\lambda}X$ holds for arbitrary $X \in \mathfrak{X}(\mathcal{A}_M)$. Then, also $E' = \frac{\partial E}{\partial\lambda}\lambda'$. For instance one could take $E = \lambda^n$.

The related canonical isomorphism $\sharp|_\lambda : T^*_\lambda\mathcal{A}_M \to T_\lambda\mathcal{A}_M$ has the form

$$\alpha^\sharp\big|_\lambda = \lambda' r(E\alpha) + E\, r^*(\lambda'\alpha). \tag{4.10}$$

We require the kernel of $\sharp$ to be trivial on $\mathcal{A}_M$.

Theorem 4.2. *Assume that $r \in \mathrm{End}\,\mathcal{A}$ satisfies (4.4) and it is invariant on $\mathcal{A}_M$. Then, the following statements are valid:*

(i) The Levi-Civita connection of the metric (4.9) *is given by*

$$\nabla_{\alpha^\sharp}\gamma = \mathrm{D}_{\alpha^\sharp}\gamma - r(E\alpha)\gamma' - \alpha\, r(E\gamma)' + \frac{\partial E}{\partial\lambda} r^*(\lambda'\alpha)\gamma, \tag{4.11}$$

where $\alpha,\gamma \in \Lambda^1(\mathcal{A}_M)$.

(ii) The metric (4.9) *is flat, that is the curvature tensor* (A.12) *vanish identically on* $\mathcal{A}_M$.

Lemma 4.3. *The identity* (4.4), *for* $r \in \operatorname{End}\mathcal{A}$, *is equivalent to*

$$r^*(r^*(a)b') - r^*(a\, r(b)') + r^*(a)r(b)' = \kappa\, ab'. \tag{4.12}$$

Proof. Define functionals in the form:

$$\begin{aligned} K_1[a,b] &:= r(r(a)b') + r(a\, r(b)') - r(a)r(b)' - \kappa\ ab', \\ K_2[a,b] &:= r^*(r^*(a)b') - r^*(a\, r(b)') + r^*(a)r(b)' - \kappa\, ab', \end{aligned} \tag{4.13}$$

which vanishing is equivalent to the identities (4.4) and (4.12). The lemma follows from the equality

$$\operatorname{Tr}(K_1[a,b]\, c) = \operatorname{Tr}(aK_2[c,b])$$

and the fact that the trace form is nondegenerate. □

Proof of Theorem 4.2. The tensor field (A.8) related to (4.11) has the form

$$\Gamma_\alpha\gamma = r(E\alpha)\gamma' + \alpha\, r(E\gamma)' - \frac{\partial E}{\partial\lambda} r^*(\lambda'\alpha)\gamma. \tag{4.14}$$

We must show that it satisfies (A.9) and (A.10). Let

$$\alpha\diamond\beta := r(E\alpha)\beta + \alpha\, r(E\beta) \quad\Longrightarrow\quad g^*(\alpha,\beta) = \operatorname{Tr}(\lambda'\alpha\diamond\beta)\,.$$

Using the relation (4.4) and properties of the trace form one can show that

$$\operatorname{Tr}(\lambda'\alpha\diamond\Gamma_\beta\gamma) = \operatorname{Tr}(\lambda'\beta\diamond\Gamma_\alpha\gamma)\,, \tag{4.15}$$

which is equivalent to (A.9).

Calculating the directional derivative of the metric (4.9) one finds the formula

$$\begin{aligned} (\mathrm{D}_{\alpha^\sharp} g^*)(\beta,\gamma) &= -\operatorname{Tr}\big(\alpha^\sharp(r(E\beta)\gamma + \beta r(E\gamma))'\big) + \operatorname{Tr}\Big(\alpha^\sharp\frac{\partial E}{\partial\lambda}(\beta r^*(\lambda'\gamma) + r(\lambda'\beta)\gamma)\Big) \\ &= -g^*(\alpha,\Gamma_\beta\gamma) - g^*(\alpha,\Gamma_\gamma\beta)\,. \end{aligned}$$

Now, the second condition (A.10) is a straightforward consequence of (A.9) and (4.15).

To calculate the curvature tensor (A.12) first we need the directional derivative of (4.14), which is given by

$$(\mathrm{D}_{\beta^\sharp}\Gamma)(\alpha,\gamma) = r\Big(\frac{\partial E}{\partial\lambda}\alpha\beta^\sharp\Big)\gamma' + \alpha\, r\Big(\frac{\partial E}{\partial\lambda}\beta^\sharp\gamma\Big)' - \frac{\partial^2 E}{\partial\lambda^2} r^*(\lambda'\alpha)\beta^\sharp\gamma - \frac{\partial E}{\partial\lambda} r^*\big(\alpha(\beta^\sharp)'\big)\,\gamma,$$

where $\beta^\sharp = \lambda' r(E\beta) + E\, r^*(\lambda'\beta) \in \mathfrak{X}(\mathcal{A}_M)$. Substituting the above formula and (4.14) to (A.12) one can show that the curvature vanish. One must use the identities (4.4) and (4.12). The calculation is straightforward, however slightly tedious, so we omit it. □

Remark 4.4. Notice that the generalised metric (4.9) and the Γ tensors (4.14) are linear in E. Hence, any two metrics (4.9) defined by different E are compatible and can generate the corresponding flat pencil.

Proposition 4.5. *For arbitrary $X \in \mathfrak{X}(\mathcal{A}_M)$ and $\gamma \in \Lambda^1(\mathcal{A}_M)$ the following formula holds:*

$$\nabla_X \gamma^\sharp \equiv (\nabla_X \gamma)^\sharp = \{\lambda, r(E\gamma)\} + E\, r^*(\{\lambda, \gamma\})\,, \tag{4.16}$$

where ∇ is the Levi-Civita connection (4.11) for the metric (4.9), $\sharp$ is the canonical isomorphism (4.10) and

$$\{a, b\} := a'\, \mathrm{D}_X b - \mathrm{D}_X a\, b' \tag{4.17}$$

is a Poisson bracket defined on the algebra $\mathcal{A}$.

Proof. Applying the canonical isomorphism (4.10) to (4.14) and using the relations (4.4) and (4.12) one finds that

$$(\Gamma_\alpha \gamma)^\sharp = \alpha^\sharp r(E\gamma)' + E r^*(\alpha^\sharp \gamma') - \lambda' r\Big(\frac{\partial E}{\partial \lambda} \alpha^\sharp \gamma\Big),$$

where $\alpha^\sharp = \lambda' r(E\alpha) + E\, r^*(\lambda'\alpha)$. Hence,

$$\begin{aligned}(\nabla_{\alpha^\sharp}\gamma)^\sharp &= (\mathrm{D}_{\alpha^\sharp}\gamma - \Gamma_\alpha \gamma)^\sharp \\ &= \lambda'\, \mathrm{D}_{\alpha^\sharp} r(E\gamma) - \alpha^\sharp r(E\gamma)' - \lambda' r^*(\lambda'\, \mathrm{D}_{\alpha^\sharp}\gamma - \alpha^\sharp \gamma').\end{aligned}$$

Now, setting $X \equiv \alpha^\sharp \in \mathfrak{X}(\mathcal{A}_M)$ and defining $\{,\} := \partial \wedge \mathrm{D}_X$ we get the formula (4.16). Notice that $\mathrm{D}_X \lambda = X$. This Poisson bracket is well-defined on $\mathcal{A}$ since it is assumed that the vector fields on $\mathcal{A}_M$ commute with the derivation ∂. □

Proposition 4.6. *The condition (4.4) is a sufficient condition for $r \in \operatorname{End}\mathcal{A}$ to be a classical r-matrix with respect to the Poisson bracket (4.17) (see Appendix C), which means that*

$$r(\{r(a), b\}) + r(\{a, r(b)\}) - \{r(a), r(b)\} = \kappa\, \{a, b\}$$

holds.

The proof is straightforward using the relation (4.4) and the assumption that $\mathrm{D}_X r = 0$.

Remark 4.7. For $E = 1$ the metric (4.9) reduces to the linear metric (4.3). The most natural choice is $E = \lambda^n$ for $n \geqslant 0$. (One can also imagine more nonstandard choices of E.) This case, through the two above propositions, corresponds to the Lie-Poisson brackets from Theorem C.1. Compare the formula (4.16) with (C.1). Under appropriate assumptions, the above formalism gives alternative proof to Theorem C.1 and it can also be considered as its generalisation.

Remark 4.8. The formula (4.16) can be useful in finding flat coordinates of the corresponding metric (4.9). To find them we can look for linearly independent flat (covariantly constant) 1-forms γ^i, that is $\nabla\gamma^i = 0$. Using the relation (4.16) it is sufficient to postulate that

$$\{\gamma^i, \lambda\} = 0 \qquad \text{and} \qquad r(E\gamma^i) = 0. \tag{4.18}$$

Then, locally we have $\gamma^i = dt^i$ and t^i constitute flat coordinates. Particularly, in this way we can obtain, taking $E = 1$, flat coordinates for the metric (4.3).

Remark 4.9. Consider the loop algebra $\mathcal{L}(\mathcal{A}_M) := \{\gamma : \mathbb{S}^1 \to \mathcal{A}_M \,|\, \gamma' = 0\}$ of loops in $\mathcal{A}_M$ such that the derivative ∂ is transversal to these loops. Then, the formula (4.17) taking $X \equiv \partial_x$ defines Poisson bracket on $\mathcal{L}(\mathcal{A}_M)$ and the classical r-matrix scheme could be applied to $\mathcal{L}(\mathcal{A}_M)$, see Appendix C.

4.3. **Frobenius structure.** The linear metric (4.3) can be written in the form

$$\eta^*_\lambda(\alpha, \beta) = \operatorname{Tr}(r^*(\lambda'\alpha)\beta + \alpha\, r^*(\lambda'\beta)),$$

which suggests that there could be defined in the cotangent bundle $T^*\mathcal{A}_M$ invariant multiplication, such that $\eta^*(\alpha, \beta) = \operatorname{Tr}(\alpha \circ \beta)$, acquiring the following form

$$\alpha \circ \beta = r^*(\lambda'\alpha)\beta + \alpha\, r^*(\lambda'\beta). \tag{4.19}$$

According to Proposition 3.1, if r^* was satisfying the Rota-Baxter identity (3.2), this multiplication would be associative. The related co-unity 1-form would be

$$\varepsilon\big(\alpha^\sharp\big) \equiv \langle \varepsilon, \alpha^\sharp \rangle = \operatorname{Tr} \alpha. \tag{4.20}$$

As we have seen in Section 2, on a Frobenius manifold the counity is necessarily closed.

Proposition 4.10. *Assume that $r \in \operatorname{End} \mathcal{A}$ satisfies (4.4) so that Theorem 4.1 holds. Then, the condition*

$$r^*(\gamma') + r(\gamma)' = 0, \tag{4.21}$$

where $\gamma \in \Lambda^1(\mathcal{A}_M)$ is arbitrary, is a sufficient condition for vanishing of $d\varepsilon$, where ε is the 1-form defined by (4.20).

Proof. The exterior derivative of a 1-form ε is

$$d\varepsilon(X, Y) = X(\varepsilon(Y)) - Y(\varepsilon(X)) - \varepsilon([\![X, Y]\!]),$$

where X, Y are vector fields. Setting $X = \alpha^\sharp$ and $Y = \beta^\sharp$ and using (A.11), it follows that for the 1-form (4.20):

$$\begin{aligned} d\varepsilon(\alpha^\sharp, \beta^\sharp) &= \operatorname{Tr}\big(\mathrm{D}_{\alpha^\sharp}\beta - \mathrm{D}_{\beta^\sharp}\alpha - \nabla_{\alpha^\sharp}\beta + \nabla_{\beta^\sharp}\alpha\big) = \operatorname{Tr}(\Gamma_\alpha\beta - \Gamma_\beta\alpha) \\ &= 2\operatorname{Tr}(\alpha(r^*(\beta') + r(\beta)')). \end{aligned}$$

Hence, the assertion follows. □

Remark 4.11. In fact, one can show that if r satisfies the relation (4.4), the condition (4.21), and the derivation ∂ is *onto* ($\operatorname{Im} \partial = \mathcal{A}$), then r^* satisfies the Rota-Baxter identity (3.2) and the multiplication (4.19) is associative. There arise question when one can define on $\mathcal{A}_M$ structure of a Frobenius manifold.

5. Construction of pre-Frobenius manifolds

The preliminary setting is the same as in the previous section. Let $\mathcal{A}$ be a commutative associative unital algebra equipped with a nondegenerate trace form $\operatorname{Tr} : \mathcal{A} \to \mathbb{K}$ and invariant derivation $\partial \in \operatorname{Der} \mathcal{A}$, that is $\operatorname{Tr} a' = 0$, where $a' \equiv \partial a$. Consider the subspace $\mathcal{A}_M \subset \mathcal{A}$, which will constitute an underlying manifold, such that ∂ is transversal to $\mathcal{A}_M$. Then, the tangent and cotangent spaces can be identified with appropriate subspaces of $\mathcal{A}$ through the trace form.

Let be given some linear map $\ell \in \operatorname{End} \mathcal{A}$, which satisfy the Rota-Baxter identity (3.2), that is

$$\ell(\ell(a)b) + \ell(a\ell(b)) - \ell(a)\ell(b) = \kappa\, ab, \tag{5.1}$$

for some $\kappa \in \mathbb{K}$. Then, for fixed $\lambda \in \mathcal{A}$ we define the second commutative multiplication in $\mathcal{A}$ by the formula

$$a \circ b := \ell(\lambda' a)b + a\ell(\lambda' b). \tag{5.2}$$

By Proposition 3.1 this multiplication is associative.

5.1. **Structure of Frobenius algebra.** We define the contravariant metric at a point $\lambda \in \mathcal{A}_M$ by the formula

$$\eta^*_\lambda(\alpha, \beta) := \operatorname{Tr}(\alpha \circ \beta) \equiv \operatorname{Tr}(\ell(\lambda'\alpha)\beta + \alpha\, \ell(\lambda'\beta)) \qquad \alpha, \beta \in T^*_\lambda \mathcal{A}_M \tag{5.3}$$

and require that on $\mathcal{A}_M$ it is nondegenerate. The related canonical isomorphism $\sharp|_\lambda : T^*_\lambda \mathcal{A}_M \to T_\lambda \mathcal{A}_M$ is

$$\alpha^\sharp\big|_\lambda = \ell(\lambda'\alpha) + \lambda'\ell^*(\alpha), \tag{5.4}$$

where ℓ^* is the adjoint of ℓ with respect to the trace form.

Suppose that the multiplication (5.2) restricts properly to $T^*_\lambda \mathcal{A}_M$. Then, the formula (5.2) defines in the cotangent bundle associative and commutative multiplication, such that

$$\circ : T^*_\lambda \mathcal{A}_M \times T^*_\lambda \mathcal{A}_M \to T^*_\lambda \mathcal{A}_M \qquad (\alpha, \beta) \mapsto \circ(\alpha, \beta) \equiv \alpha \circ \beta. \tag{5.5}$$

This multiplication is invariant with respect to the metric (5.3). Hence, the contravariant metric (5.3) and the multiplication (5.2), if it is unital, define the structure of Frobenius algebra in the cotangent bundle $T^*\mathcal{A}_M$.

Remark 5.1. In practice the multiplication (5.2) does not have to naturally restrict to $T^*_\lambda \mathcal{A}_M$. If the complement $(T^*_\lambda \mathcal{A}_M)^c = \mathcal{A} \setminus T^*_\lambda \mathcal{A}_M$ is an ideal in $\mathcal{A}_M$ with respect to the multiplication (5.2), then we can define multiplication in $T^*_\lambda \mathcal{A}_M$ by means of the quotient algebra $\mathcal{A}/(T^*_\lambda \mathcal{A}_M)^c$. Notice that $T^*_\lambda \mathcal{A}_M \cong \mathcal{A}/(T^*_\lambda \mathcal{A}_M)^c$. In this case, we must require that $(T^*_\lambda \mathcal{A}_M)^c \circ \mathcal{A} \subset (T^*_\lambda \mathcal{A}_M)^c$. Still we need the metric (5.3) to be compatible with the quotient structure. Thus, additionally we must require that

$$\eta^*_\lambda((T^*_\lambda \mathcal{A}_M)^c, \mathcal{A}) = \operatorname{Tr}((T^*_\lambda \mathcal{A}_M)^c \circ \mathcal{A}) = 0.$$

Then, the metric (5.3) is invariant with respect to the multiplication defined by the quotient structure.

Remark 5.2. Assume that the multiplication (5.5) is unital and the unity 1-form is given by ε. Then, for arbitrary $\alpha \in \Lambda^1(\mathcal{A}_M)$ we have $\langle \varepsilon, \alpha^\sharp \rangle = \operatorname{Tr}(\varepsilon \circ \alpha) = \operatorname{Tr} \alpha$. On the other hand $\langle \varepsilon, \alpha^\sharp \rangle = \langle \alpha, e \rangle \equiv \operatorname{Tr}(e\alpha)$, where $e = \varepsilon^\sharp$ is the unity vector field. Since the trace is nondegenerate, we see that for the multiplication (5.5) the unit vector field $e = 1$, that is e coincides with the unity 1 of the original algebra $\mathcal{A}$ or lies in the same equivalence class. Thus, to have the Frobenius algebra structure in the cotangent bundle $T^*\mathcal{A}_M$ we must require that $e = 1 \in T_\lambda \mathcal{A}_M$.

5.2. **Main theorem.** The metric (5.3) can be written in the form

$$\eta^*(\alpha, \beta) := \operatorname{Tr}(\lambda' \ell^*(\alpha)\beta + \lambda' \alpha \ell^*(\beta)),$$

which, when $r = \ell^*$, coincides with (4.3). By Theorem 4.1 the sufficient condition for flatness of the metric is the identity (4.4). It turns out, that for $r = \ell^*$ the condition (4.4) is fulfilled, if the Rota-Baxter identity together with (4.21) hold.

Lemma 5.3. *If $\ell \in \operatorname{End} \mathcal{A}$ satisfies the Rot-Baxter identity (5.1) and the relation*

$$\ell(a') + \ell^*(a)' = 0, \tag{5.6}$$

then $r = \ell^*$ fulfils the identity (4.4), that is

$$\ell^*(\ell^*(a)b') + \ell^*(a\ell^*(b)') - \ell^*(a)\ell^*(b)' = \kappa\, ab',$$

and also

$$\ell(\ell^*(a)b') - \ell(a\ell(b)') - \ell^*(a)\ell(b)' = \kappa\, ab'. \tag{5.7}$$

Proof. Let

$$\begin{aligned} \widetilde{K}_1[a,b] &:= \ell(\ell(a)b) + \ell(a\ell(b)) - \ell(a)\ell(b) - \kappa\, ab, \\ \widetilde{K}_2[a,b] &:= \ell(\ell^*(a)b') - \ell(a\ell(b)') - \ell^*(a)\ell(b)' + \kappa\, ab', \end{aligned}$$

which are connected with (5.1) and (5.7). Then,

$$\begin{aligned} \operatorname{Tr}\big(\widetilde{K}_1[a,b']\, c\big) &= \operatorname{Tr}\big(aK_1[c,b]\big) \\ \operatorname{Tr}\big(\widetilde{K}_2[a,b]\, c\big) &= \operatorname{Tr}\big(bK_1[a,c] - bK_1[c,a]\big), \end{aligned}$$

where K_1 is given by (4.13) for $r = \ell^*$. Now, the results of the lemma follows from the nondegeneracy of the trace form. □

We will show that under certain technical assumption on a submanifold $\mathcal{A}_M$ of $\mathcal{A}$ we can define structure of a pre-Frobenius manifold.

Theorem 5.4. *Let $\ell \in \operatorname{End} \mathcal{A}$ be invariant on $\mathcal{A}_M$ and satisfy the Rota-Baxter identity (5.1) and the requirement (5.6). Then, the following statements hold:*

(i) The Levi-Civita connection for the contravariant metric (5.3) has the form

$$\nabla_{\alpha^\sharp}\gamma = \mathrm{D}_{\alpha^\sharp}\gamma + \alpha\, \ell(\gamma') - \ell^*(\alpha)\gamma'. \tag{5.8}$$

(ii) The metric (5.3) is flat, that is the curvature tensor $R(\alpha^\sharp, \beta^\sharp)\gamma$ vanish identically on $\mathcal{A}_M$.

(iii) The co-unity 1-form ε, such that $\varepsilon(\alpha^\sharp) \equiv \operatorname{Tr} \alpha$, is closed.

(iv) The tensor $\nabla *$ is symmetric in all three arguments, where

$$\alpha^\sharp * \beta^\sharp := (\alpha \circ \beta)^\sharp \qquad \alpha, \beta \in \Lambda^1(\mathcal{A}_M)$$

is the induced multiplication in the tangent bundle $T\mathcal{A}_M$. In principle, the relation

$$(\nabla_{\alpha^\sharp} \circ)\,(\beta, \gamma) = \big(\nabla_{\beta^\sharp} \circ\big)\,(\alpha, \gamma) \tag{5.9}$$

is valid.

Proof. Assuming that $r = \ell^*$, the first three points of the theorem are straightforward consequence of Lemma 5.3, Theorem 4.1 and Proposition 4.10.

It is left to show that $\nabla *$ is symmetric in all its arguments, i.e.

$$(\nabla_{\alpha^\sharp} *)\,(\beta^\sharp, \gamma^\sharp) = \big(\nabla_{\beta^\sharp} *\big)\,(\alpha^\sharp, \gamma^\sharp).$$

Since ∇ is the Levi-Civita connection the following relation is valid:

$$(\nabla_{\alpha^\sharp} *)\,(\beta^\sharp, \gamma^\sharp) = ((\nabla_{\alpha^\sharp} \circ)\,(\beta, \gamma))^\sharp\,.$$

Hence, it is sufficient to show that (5.9) holds.

Expanding $\nabla\circ$, one finds that

$$\begin{aligned}(\nabla_{\alpha^\sharp} \circ)\,(\beta, \gamma) &= \nabla_{\alpha^\sharp}(\beta \circ \gamma) - \nabla_{\alpha^\sharp}\beta \circ \gamma - \beta \circ \nabla_{\alpha^\sharp}\gamma \\ &= (\mathrm{D}_{\alpha^\sharp} \circ)(\beta, \gamma) - \Gamma_\alpha(\beta \circ \gamma) + \Gamma_\alpha \beta \circ \gamma + \beta \circ \Gamma_\alpha \gamma,\end{aligned} \tag{5.10}$$

where Γ is the tensor field (A.8), which by (5.8) has the form $\Gamma_\alpha \gamma = \ell^*(\alpha)\gamma' - \alpha\,\ell(\gamma')$. Hence,

$$\begin{aligned}\Gamma_\alpha(\beta \circ \gamma) &= \ell^*(\alpha)(\beta \circ \gamma)' - \alpha\ell((\beta \circ \gamma)') \\ &= \ell^*(\alpha)\ell(\lambda'\beta)'\gamma + \ell^*(\alpha)\ell(\lambda'\beta)\gamma' + \ell^*(\alpha)\beta'\ell(\lambda'\gamma) + \ell^*(\alpha)\beta\ell(\lambda'\gamma)' \\ &\quad - \alpha\ell(\ell(\lambda'\beta)'\gamma) - \alpha\ell(\ell(\lambda'\beta)\gamma') - \alpha\ell(\beta'\ell(\lambda'\gamma)) - \alpha\ell(\beta\ell(\lambda'\gamma)')\end{aligned}$$

and

$$\begin{aligned}\Gamma_\alpha\beta \circ \gamma &= \ell(\lambda'\Gamma_\alpha\beta)\,\gamma + \Gamma_\alpha\beta\ell(\lambda'\gamma) \\ &= \ell(\ell^*(\alpha)\lambda'\beta')\,\gamma - \ell(\lambda'\alpha\ell(\beta'))\,\gamma + \ell^*(\alpha)\beta'\ell(\lambda'\gamma) - \alpha\ell(\beta')\ell(\lambda'\gamma).\end{aligned}$$

Remind that $\mathrm{D}_{\alpha^\sharp}\lambda' = (\alpha^\sharp)'$, where $\alpha^\sharp \in \mathfrak{X}(\mathcal{A}_M)$ is given by (5.4). Thus,

$$\begin{aligned}(\mathrm{D}_{\alpha^\sharp} \circ)(\beta, \gamma) &= \ell(\mathrm{D}_{\alpha^\sharp}\lambda'\beta)\,\gamma + \beta\ell(\mathrm{D}_{\alpha^\sharp}\lambda'\gamma) = \ell\big(\alpha^{\sharp\prime}\beta\big)\,\gamma + \beta\ell\big(\alpha^{\sharp\prime}\gamma\big) \\ &= \ell(\ell^*(\alpha)\lambda''\beta)\,\gamma - \ell(\ell(\alpha')\lambda'\beta)\,\gamma + \ell(\ell(\lambda'\alpha)'\beta)\,\gamma \\ &\quad + \beta\ell(\ell^*(\alpha)\lambda''\gamma) - \beta\ell(\ell(\alpha')\lambda'\gamma) + \beta\ell(\ell(\lambda'\alpha)'\gamma)\,.\end{aligned}$$

Now, substituting the above formulae to (5.10) we have

$$
\begin{aligned}
&(\nabla_{\alpha^\sharp}\circ)(\beta,\gamma) - \big(\nabla_{\beta^\sharp}\circ\big)(\alpha,\gamma) = (\nabla_{\alpha^\sharp}\circ)(\beta,\gamma) - \{\alpha \leftrightarrow \beta\}\\
&\quad = \ell(\ell^*(\alpha)\lambda''\beta)\,\gamma - \cancel{\ell(\ell(\alpha')\lambda'\beta)\,\gamma} + \ell(\ell(\lambda'\alpha)'\beta)\,\gamma + \beta\ell(\ell^*(\alpha)\lambda''\gamma)\\
&\qquad - \beta\ell(\ell(\alpha')\lambda'\gamma) + \cancel{\beta\ell(\ell(\lambda'\alpha)'\gamma)} - \ell^*(\alpha)\ell(\lambda'\beta)'\gamma - \cancel{\ell^*(\alpha)\ell(\lambda'\beta)\gamma'}\\
&\qquad - \cancel{\ell^*(\alpha)\beta'\ell(\lambda'\gamma)} - \ell^*(\alpha)\beta\ell(\lambda'\gamma)' + \cancel{\alpha\ell(\ell(\lambda'\beta)'\gamma)} + \alpha\ell(\ell(\lambda'\beta)\gamma')\\
&\qquad + \alpha\ell(\beta'\ell(\lambda'\gamma)) + \alpha\ell(\beta\ell(\lambda'\gamma)') + \ell(\ell^*(\alpha)\lambda'\beta')\,\gamma - \cancel{\ell(\lambda'\alpha\ell(\beta'))\,\gamma}\\
&\qquad + \cancel{\ell^*(\alpha)\beta'\ell(\lambda'\gamma)} - \alpha\ell(\beta')\ell(\lambda'\gamma) + \beta\ell(\ell^*(\alpha)\lambda'\gamma') - \beta\ell(\lambda'\alpha\ell(\gamma'))\\
&\qquad + \cancel{\ell^*(\alpha)\ell(\lambda'\beta)\gamma'} - \alpha\ell(\lambda'\beta)\ell(\gamma') - \{\alpha \leftrightarrow \beta\},
\end{aligned}
$$

where $\{\alpha \leftrightarrow \beta\}$ stands for all the remaining terms arising by permutation of α with β in the preceding terms. Some terms cancel out as in the preceding terms we are allowed to permute α and β with simultaneous change of sign. Using that property we can assort all terms obtaining

$$
\begin{aligned}
&(\nabla_{\alpha^\sharp}\circ)(\beta,\gamma) - \big(\nabla_{\beta^\sharp}\circ\big)(\alpha,\gamma) =\\
&\quad = [\ell(\ell^*(\alpha)(\lambda'\beta)') - \ell(\alpha\ell(\lambda'\beta)') - \ell^*(\alpha)\ell(\lambda'\beta)']\,\gamma\\
&\qquad + \beta[\ell(\ell^*(\alpha)(\lambda'\gamma)') - \ell(\alpha\ell(\lambda'\gamma)') - \ell^*(\alpha)\ell(\lambda'\gamma)']\\
&\qquad - \beta[\ell(\ell(\alpha')\lambda'\gamma) + \ell(\alpha'\ell(\lambda'\gamma)) - \ell(\alpha')\ell(\lambda'\gamma)]\\
&\qquad + \alpha[\ell(\ell(\lambda'\beta)\gamma') + \ell(\lambda'\beta\ell(\gamma')) - \ell(\lambda'\beta)\ell(\gamma')] - \{\alpha \leftrightarrow \beta\}\\
&\quad = -\kappa\,\alpha(\lambda'\beta)'\gamma - \kappa\,\alpha\beta(\lambda'\gamma)' - \kappa\,\lambda'\alpha'\beta\gamma + \kappa\,\lambda'\alpha\beta\gamma' - \{\alpha \leftrightarrow \beta\} = 0,
\end{aligned}
$$

where the identities (5.7) and (5.1) were used. This completes the proof. □

Remark 5.5. The above proof and formalism is much more basic if the endomorphism ℓ is antisymmetric ($\ell^* = -\ell$) and commutes with the derivation ∂, equivalently $\ell' = 0$. In this situation the proof of Theorem 5.4 is more straightforward.

5.3. **The recurrence formula.** For derivation of the pre-potential on a Frobenius manifold using Proposition 2.4 we must see how the recurrence formula (2.12) fits in the above scheme.

Proposition 5.6. *Assume that* $d\mathcal{H}^k_{(n)} : \mathcal{A}_M \to \mathcal{A}$ *are differentiable functions of the variable* $\lambda \in \mathcal{A}_M$ *such that*

$$
\big(d\mathcal{H}^k_{(n)}\big)' = \frac{\partial d\mathcal{H}^k_{(n)}}{\partial\lambda}\lambda' \qquad \textit{and} \qquad \mathrm{D}_X d\mathcal{H}^k_{(n)} = \frac{\partial d\mathcal{H}^k_{(n)}}{\partial\lambda}X,
$$

where $X \in \mathfrak{X}(\mathcal{A}_M)$. *Then, the recurrence formula* (2.12) *takes the form*

$$
\frac{\partial d\mathcal{H}^k_{(n)}}{\partial\lambda} = d\mathcal{H}^k_{(n-1)}. \tag{5.11}
$$

Proof. From the assumptions and formulae (5.2) and (5.8) we have

$$
\begin{aligned}
\nabla_{\alpha^\sharp} d\mathcal{H}^k_{(n)} &= \frac{\partial d\mathcal{H}^k_{(n)}}{\partial\lambda}\alpha^\sharp + \alpha\,\ell\big((d\mathcal{H}^k_{(n)})'\big) - \ell^*(\alpha)(d\mathcal{H}^k_{(n)})'\\
&= \ell(\lambda'\alpha)\frac{\partial d\mathcal{H}^k_{(n)}}{\partial\lambda} + \alpha\,\ell\Big(\lambda'\frac{\partial d\mathcal{H}^k_{(n)}}{\partial\lambda}\Big)
\end{aligned}
$$

and

$$\alpha \circ d\mathcal{H}^k_{(n-1)} = \ell(\lambda'\alpha) d\mathcal{H}^k_{(n-1)} + \alpha\, \ell(\lambda' d\mathcal{H}^k_{(n-1)}),$$

which substituted to (2.12) give (5.11). □

Proposition 5.7. *For arbitrary $\alpha, \beta \in \Lambda^1(\mathcal{A}_M)$ the following relation is valid:*

$$(\alpha \circ \beta)^\sharp = \alpha^\sharp \ell(\lambda'\beta) + \lambda' \ell^*(\alpha^\sharp \beta), \tag{5.12}$$

where $\alpha^\sharp \in \mathfrak{X}(\mathcal{A}_M)$ is given by (5.4).

The proof is straightforward using the identities (5.1) and (3.4).

6. Frobenius manifolds on the space of meromorphic functions

6.1. **Algebra of meromorphic functions.** Our aim is to illustrate the scheme for construction of Frobenius manifolds by applying it to the algebra of meromorphic functions on the Riemann sphere $\mathbb{CP}^1$. So, let

$$\mathcal{A} = \{ f : \mathbb{C}^* \mapsto \mathbb{C} \,|\, f \text{ is meromorphic} \},$$

where $\mathbb{C}^* \cong \mathbb{CP}^1$ is the extended complex plane. Let p denote the variable in $\mathbb{C}^*$. The space $\mathcal{A}$ is an infinite dimensional algebra with respect to the commutative associative multiplication of functions.

We define the derivation $\partial \in \operatorname{Der}\mathcal{A}$ for $s = 0$ or $s = 1$ by the formula

$$f' \equiv \partial f := p^s \frac{\partial f}{\partial p} \qquad f \in \mathcal{A}. \tag{6.1}$$

We will consider this two different cases of $s = 0$ and $s = 1$ simultaneously.

Our aim is to define Frobenius algebras taking advantage of expansions near some marked points $\nu \in \mathbb{C}^*$ on the extended complex plane. We will distinguish three types of them. The first one is a fixed point at infinity, i.e. $\nu = \infty$. The second one is a fixed finite point, without loss of generality we take $\nu = 0$. The last one is a 'finite' not fixed point, which can vary over the complex plane, i.e. $\nu = v \in \mathbb{C}$. Important is fact that v will be taken as one of the coordinates on the underlying manifold.

At a given marked point ν the trace form $\operatorname{Tr}_\nu : \mathcal{A} \to \mathbb{C}$ is defined by

$$\operatorname{Tr}_\nu(f) := \epsilon \operatorname{res}_{p=\nu}\big(p^{-s} f\big) \qquad s = 0, 1, \tag{6.2}$$

where $\epsilon = -1$ for $\nu = \infty$ and $\epsilon = 1$ otherwise. The invariance of (6.2) with respect to the derivation (6.1) is a straightforward consequence of the integration by parts.

6.2. **The endomorphisms.** Meromorphic functions can be expanded into Laurent series at the marked points and at these points we can define projections on the finite parts of the series. Therefore, for any $f \in \mathcal{A}$ at $\nu = \infty$ let

$$[f(p)]^\infty_{\geqslant l} := \sum_{i \geqslant l} a_i p^i \qquad f(p) = \sum_{i \leqslant n} a_i p^i,$$

where $n = \deg_\infty f$ and $\operatorname{res}_{p=\infty} f = -a_{-1}$. For finite $\nu = 0, v$:

$$[f(p)]^\nu_{<l} := \sum_{i<l} a_i (p-\nu)^i \qquad f(p) = \sum_{i\geqslant -m} a_i (p-\nu)^i,$$

where $\deg_\nu f = m$ and $\operatorname{res}_{p=\nu} f = a_{-1}$. We will denote by $\mathcal{A}^\infty = \{\sum_i a_i p^i\}$ for $\nu = \infty$ and by $\mathcal{A}^\nu = \{\sum_i a_i (p - \nu^i)\}$ for finite ν the corresponding spaces of formal Laurent series.

We define $\ell \in \operatorname{End}\mathcal{A}$ by the formula

$$\ell(f) = p^s\left[p^{-s}f\right]^\nu_{\geqslant 0} - \frac{1}{2}f = \frac{1}{2}f - p^s\left[p^{-s}f\right]^\nu_{<0} \qquad s = 0, 1. \tag{6.3}$$

Proposition 6.1. *The endomorphisms (6.3) satisfy the Rota-Baxter identity (5.1) for $\kappa = \frac{1}{4}$. The adjoints of (6.3) with respect to the traces (6.2) have the form*

$$\ell^*(f) = \frac{1}{2}f - [f]^\nu_{\geqslant 0} = [f]^\nu_{<0} - \frac{1}{2}f \qquad f \in \mathcal{A}. \tag{6.4}$$

Moreover, the relation (5.6) is satisfied by endomorphisms (6.3).

Proof. For $s = 0$ and $s = 1$ the spaces $\mathcal{A}^\nu$ can be decomposed into direct sums of subalgebras, that is $\mathcal{A}^\nu = \mathcal{A}^\nu_{\geqslant s} \oplus \mathcal{A}^\nu_{<s}$. Hence, by Proposition 3.4 the linear maps

$$\widetilde{\ell}(f) = [f]^\nu_{\geqslant s} - \frac{1}{2}f = \frac{1}{2}f - [f]^\nu_{<s} \qquad f \in \mathcal{A}^\nu$$

satisfy the identity (3.2) and (5.1). These maps coincide with (6.3) for all cases but one.

The exception occurs for finite $\nu = v$ and $s = 1$. It must be considered separately. In this case (6.3) can be written in the form

$$\ell(f) = [f]^v_{\geqslant 1} - \frac{1}{2}f + v\left[p^{-1}f\right]^v_0 \equiv \widetilde{\ell}(f) + P(f), \tag{6.5}$$

where $[\sum_i a_i (p-v)^i]^v_0 := a_0$ and $P(f) = v[p^{-1}f]^v_0$.

Assume that $\widetilde{\ell}$ satisfies (5.1). Let P be such that $\widetilde{\ell}(P(f)g) = P(f)\widetilde{\ell}(f)$ and $P(P(f)g) = P(f)P(g)$. These requirements are satisfied by (6.5). Then, $\ell = \widetilde{\ell} + P$ satisfies (5.1) for the same κ iff

$$P\Big(\widetilde{\ell}(f)g\Big) + P\Big(f\widetilde{\ell}(g)\Big) + P(f)P(g) = 0. \tag{6.6}$$

The identity (6.6), for the map (6.5), can be showed using the following relations:

$$\left[p^{-1}[f]^v_{\geqslant 1}\, g\right]^v_0 = \left[f\left[p^{-1}g\right]^v_{<0}\right]^v_0$$
$$\left[p^{-1}f[g]^v_{\geqslant 1}\right]^v_0 = \left[f\left[p^{-1}g\right]^v_{\geqslant 0}\right]^v_0 - v\left[p^{-1}f\right]^v_0\left[p^{-1}g\right]^v_0.$$

To find ℓ^* it is sufficient to observe that

$$\operatorname{Tr}_\nu\Big(p^s\left[p^{-s}f\right]^\nu_{\geqslant 0} g\Big) = \operatorname{res}_{p=\nu}\Big(\left[p^{-s}f\right]^\nu_{\geqslant 0} g\Big) = \operatorname{res}_{p=\nu}\left(p^{-s}f[g]^\nu_{<0}\right) = \operatorname{Tr}_\nu\left(f[g]^\nu_{<0}\right).$$

The last statement of the proposition is straightforward as

$$\ell(f') + \ell^*(f)' = p^s\left[\frac{\partial f}{\partial p}\right]^\nu_{\geqslant 0} - p^s\frac{\partial[f]^\nu_{\geqslant 0}}{\partial p} = 0.$$

This finishes the proof. ∎

Remark 6.2. The construction of Frobenius manifolds in this section corresponds to the construction of bi-Hamiltonian structures for dispersionless systems with meromorphic Lax representations presented in [34]. where the classical r-matrix formalism is applied to commutative algebras equipped with Poisson bracket $\{\cdot,\cdot\} = p^s\partial_p \wedge \partial_x$. The related classical r-matrices have the form $r = P^{\nu}_{\geqslant k-s} - \frac{1}{2}$ and are classified with respect to $s, k \in \mathbb{Z}$. The condition (5.6) is crucial for Theorem 5.4 to hold. The only relevant cases of r satisfying (5.6) with $\ell = r^*$, up to equivalence, are for $s = k = 0$ and $s = k = 1$, which are related to dispersionless systems of the KdV and Toda type, respectively.

6.3. **Frobenius structure on the spaces of formal Laurent series.** First we will show how to construct Frobenius algebras and formal pre-Frobenius manifolds on appropriate submanifolds $\mathcal{A}^{\nu}_M$ of the spaces of formal Laurent series. Only then it will be natural to extend (reduce) this formalism to underlying manifolds made of meromorphic functions.

We will consider the situation when $\mathcal{A}^{\nu}_M$ consist of formal Laurent series $\lambda(p)$ at ν with prescribed order and form. For $\nu = \infty$ we define

$$\mathcal{A}^{\infty}_M = \left\{\lambda(p) = p^n + u_{n-1}p^{n-1} + u_{n-2}p^{n-2} + \dots \,|\, u_i \in \mathbb{C}, u_{n-1} = 0 \text{ for } s = 0\right\}, \tag{6.7}$$

where the degree $\deg_\infty \lambda = n$ is fixed and for finite ν we define

$$\mathcal{A}^{\nu}_M = \left\{\lambda(p) = \dots + u_{1-m}(p-\nu)^{1-m} + u_{-m}(p-\nu)^{-m} \,|\, u_i \in \mathbb{C} \text{ and } v \in \mathbb{C} \text{ (if } \nu = v)\right\},$$

where $\deg_\nu = m$ is also fixed. The complex coefficients u_i and v (when $\nu = v$) are coordinates on the underlying 'infinite-dimensional' manifolds associated to $\mathcal{A}^{\nu}_M$. The tangent spaces are spanned by derivations of $\lambda(p)$ with respect to these coordinates, that is

$$T_\lambda \mathcal{A}^{\nu}_M = \operatorname{span}\left\{\frac{\partial\lambda}{\partial u_i}, \frac{\partial\lambda}{\partial v} \text{ (if } \nu = v)\right\}$$

at point $\lambda \in \mathcal{A}^{\nu}_M$. The cotangent spaces $T^*_\lambda \mathcal{A}^{\nu}_M$ are dual spaces with respect to the corresponding trace forms (6.2).

One finds that for $\nu = \infty$:

$$T_\lambda \mathcal{A}^{\infty}_M \cong \mathcal{A}^{\infty}_{\leqslant n+s-2} \quad \Rightarrow \quad T^*_\lambda \mathcal{A}^{\infty}_M \cong \mathcal{A}^{\infty}_{\geqslant 1-n},$$

for $\nu = 0$:

$$T_\lambda \mathcal{A}^{0}_M \cong \mathcal{A}^{0}_{\geqslant -m} \quad \Rightarrow \quad T^*_\lambda \mathcal{A}^{0}_M \cong \mathcal{A}^{0}_{<m+s},$$

and for $\nu = v$:

$$T_\lambda \mathcal{A}^{v}_M \cong \begin{cases} \mathcal{A}^{v}_{\geqslant -m-1} & \text{for } m \neq 0 \\ \mathcal{A}^{v}_{\geqslant 0} & \text{for } m = 0 \end{cases} \quad \Rightarrow \quad T^*_\lambda \mathcal{A}^{v}_M \cong \begin{cases} \mathcal{A}^{v}_{\leqslant m} & \text{for } m \neq 0 \\ \mathcal{A}^{v}_{<0} & \text{for } m = 0. \end{cases}$$

Using (6.3), the multiplication (5.2) in $\mathcal{A}^{\nu}$ takes the form

$$\begin{aligned} \alpha \circ \beta &= p^s[\lambda_p\alpha]^{\nu}_{\geqslant 0}\,\beta + p^s\alpha[\lambda_p\beta]^{\nu}_{\geqslant 0} - p^s\lambda_p\alpha\beta \\ &= p^s\lambda_p\alpha\beta - p^s[\lambda_p\alpha]^{\nu}_{<0}\,\beta - p^s\alpha[\lambda_p\beta]^{\nu}_{<0}\,. \end{aligned} \tag{6.8}$$

This product will be used to define the Frobenius multiplication in the cotangent bundle $T\mathcal{A}^{\nu}_M$.

On each $\mathcal{A}^{\nu}_{M}$ the respective contravariant metric is given by the formula (5.3), that is

$$\eta^*(\alpha,\beta) = \operatorname{Tr}_\nu(\alpha\circ\beta) = \operatorname{Tr}_\nu\big(\alpha^\sharp\beta\big) \qquad \alpha,\beta\in\Lambda^1(\mathcal{A}^{\nu}_{M}). \tag{6.9}$$

The related canonical isomorphism $\sharp : T^*_\lambda\mathcal{A}^{\nu}_{M} \to T_\lambda\mathcal{A}^{\nu}_{M}$ at $\lambda\in\mathcal{A}^{\nu}_{M}$ is such that

$$\alpha^\sharp = p^s[\lambda_p\alpha]^{\nu}_{\geqslant 0} - p^s\lambda_p[\alpha]^{\nu}_{\geqslant 0} = p^s\lambda_p[\alpha]^{\nu}_{<0} - p^s[\lambda_p\alpha]^{\nu}_{<0}\,. \tag{6.10}$$

Proposition 6.3. *The metric (6.9), at a generic point $\lambda\in\mathcal{A}^{\nu}_{M}$, is nondegenerate:*

- *for $\nu=\infty$ and $s=0,1$ if $\deg_\infty\lambda\geqslant 1$;*
- *for $\nu=0$ and $s=0$ if $\deg_0\lambda=0$;*
- *for $\nu=0$ and $s=1$ if $\deg_0\lambda\geqslant 1$ or $\deg_0\lambda=-1$;*
- *for finite $\nu=v$ and $s=0,1$ if $\deg_v\lambda\geqslant -1$.*

In the remaining cases the metric is always degenerate or not well defined on the submanifold $\mathcal{A}^{\nu}_{M}$.

Proof. Let $\lambda\in\mathcal{A}^{\infty}_{M}$, then $\lambda_p\in\mathcal{A}^{\infty}_{\leqslant n-1}$. Thus for $\nu=\infty$ the map (6.10) has the image:

$$\operatorname{Im}\sharp\subset\mathcal{A}^{\infty}_{\leqslant n+s-2}\oplus\mathcal{A}^{\infty}_{\leqslant s-1}.$$

Similarly, for $\lambda\in\mathcal{A}^{0}_{M}$ we have $\lambda_p\in\mathcal{A}^{0}_{\geqslant -m-1}$ for $m\neq 0$ and $\lambda_p\in\mathcal{A}^{0}_{\geqslant 1}$ for $m=0$. Hence, for $\nu=0$

$$\operatorname{Im}\sharp\subset\begin{cases}\mathcal{A}^{0}_{\geqslant s}\oplus\mathcal{A}^{0}_{\geqslant -m+s-1} & \text{for } m\neq 0\\ \mathcal{A}^{0}_{\geqslant s} & \text{for } m=0.\end{cases}$$

For $\lambda\in\mathcal{A}^{v}_{M}$ we have $\lambda_p\in\mathcal{A}^{v}_{\geqslant -m-1}$ for $m\neq 0$ and $\lambda_p\in\mathcal{A}^{v}_{\geqslant 0}$ for $m=0$. Since $p^s\in\mathcal{A}^{v}_{\geqslant 0}$, for finite $\nu=v$ we have

$$\operatorname{Im}\sharp\subset\begin{cases}\mathcal{A}^{v}_{\geqslant 0}\oplus\mathcal{A}^{v}_{\geqslant -m-1} & \text{for } m\neq 0\\ \mathcal{A}^{v}_{\geqslant 0} & \text{for } m=0.\end{cases}$$

Since we must require that $\operatorname{Im}\sharp = T_\lambda\mathcal{A}^{\nu}_{M}$, we get the conditions on the degrees of $\lambda\in\mathcal{A}^{\nu}_{M}$. Besides, in all these cases $\ker\sharp=\emptyset$ at a generic point $\lambda\in\mathcal{A}^{\nu}_{M}$. □

Proposition 6.4. *The complement $(T^*_\lambda\mathcal{A}^{\nu}_{M})^c\equiv\mathcal{A}^{\nu}\setminus T^*_\lambda\mathcal{A}^{\nu}_{M}$ is an ideal in $\mathcal{A}^{\nu}$ with respect to (6.8) and the respective metric (6.9) is compatible with the structure of quotient algebra $\mathcal{A}^{\nu}/(T^*_\lambda\mathcal{A}^{\nu}_{M})^c$:*

- *for $\nu=\infty$ and $s=0,1$ if $\deg_\infty\lambda\geqslant 1$;*
- *for $\nu=0$ and $s=0$ if $\deg_0\lambda=0$;*
- *for $\nu=0$ and $s=1$ if $\deg_0\lambda\geqslant -1$;*
- *for finite $\nu=v$ and $s=0,1$ if $\deg_v\lambda\geqslant -1$.*

Besides, we have the natural isomorphism $T^*_\lambda\mathcal{A}^{\nu}_{M}\cong\mathcal{A}^{\nu}/(T^*_\lambda\mathcal{A}^{\nu}_{M})^c$.

Proof. Each complement is an ideal if $(T^*_\lambda\mathcal{A}^{\nu}_{M})^c\circ\mathcal{A}^{\nu}\subset(T^*_\lambda\mathcal{A}^{\nu}_{M})^c$ and the metric (6.9) is compatible with the quotient structure if $\eta^*((T^*_\lambda\mathcal{A}^{\nu}_{M})^c,\mathcal{A}^{\nu}) = \operatorname{Tr}_\nu((T^*_\lambda\mathcal{A}^{\nu}_{M})^c\circ\mathcal{A}^{\nu}) = 0$.

For $\nu=\infty$ we have $(T^*_\lambda\mathcal{A}^{\infty}_{M})^c=\mathcal{A}^{\infty}_{\leqslant -n}$. Since $\lambda_p\in\mathcal{A}^{\infty}_{<n}$ it follows that $\big[\lambda_p\mathcal{A}^{\infty}_{\leqslant -n}\big]^{\infty}_{<0} = \lambda_p\mathcal{A}^{\infty}_{\leqslant -n}$. Hence, for arbitrary $f\in\mathcal{A}^{\infty}$ and $s=0$ or $s=1$:

$$f\circ\mathcal{A}^{\infty}_{\leqslant -n} = -p^s[\lambda_p f]^{\infty}_{<0}\,\mathcal{A}^{\infty}_{\leqslant -n}\subset\mathcal{A}^{\infty}_{<s-n}\subset\mathcal{A}^{\infty}_{\leqslant -n}.$$

Thus, $\mathcal{A}^{\infty}_{\leqslant -n}$ is an ideal. Besides,

$$\operatorname{Tr}_{\infty}\big(f\circ \mathcal{A}^{\infty}_{\leqslant -n}\big) = -\operatorname{res}_{p=\infty}\mathcal{A}^{\infty}_{<-n} = 0 \qquad\Longleftrightarrow\qquad n\geqslant 1.$$

Hence the respective metric (6.9) is compatible with the quotient structure $\mathcal{A}^{\nu}/\mathcal{A}^{\infty}_{\leqslant -n}$.

For $\nu=0$ we have $(T^*_{\lambda}\mathcal{A}^0_M)^c = \mathcal{A}^0_{\geqslant m+s}$, $\lambda_p\in\mathcal{A}^0_{\geqslant -m-1}$ for $m\neq 0$ and $\lambda_p\in\mathcal{A}^0_{\geqslant 0}$ for $m=0$. Thus,

$$\big[\lambda_p\mathcal{A}^0_{\geqslant m+s}\big]^0_{\geqslant 0} = \begin{cases}\lambda_p\mathcal{A}^0_{\geqslant m+s} + (s-1)\big[\lambda_p\mathcal{A}^0_{\geqslant m+s}\big]^0_{-1}p^{-1} & \text{for } m\neq 0\\ \lambda_p\mathcal{A}^0_{\geqslant s} & \text{for } m=0\end{cases}$$

and consequently for arbitrary $f\in\mathcal{A}^0$:

$$f\circ\mathcal{A}^0_{\geqslant m+s} = \begin{cases}p^s[\lambda_p f]^0_{\geqslant 0}\,\mathcal{A}^0_{\geqslant m+s} + (s-1)p^{s-1}f\big[\lambda_p\mathcal{A}^0_{\geqslant m+s}\big]^0_{-1} & \text{for } m\neq 0\\ p^s[\lambda_p f]^0_{\geqslant 0}\,\mathcal{A}^0_{\geqslant s} & \text{for } m=0.\end{cases}$$

Hence, $(T^*_{\lambda}\mathcal{A}^0_M)^c$ is an ideal for $s=0$ only if $m=0$ and for $s=1$ if $m\geqslant -1$. With respect to the metric (6.9) we have

$$\operatorname{Tr}_0\big(f\circ\mathcal{A}^0_{\geqslant m+s}\big) = \operatorname{res}_{p=0}\mathcal{A}^0_{\geqslant m+s} = 0 \qquad\Longleftrightarrow\qquad m\geqslant -s.$$

For $\nu=v$ the complement $(T^*_{\lambda}\mathcal{A}^v_M)^c = \mathcal{A}^v_{>m}$ for $m\neq 0$ and $(T^*_{\lambda}\mathcal{A}^v_M)^c = \mathcal{A}^v_{\geqslant 0}$ for $m=0$. Since $\lambda_p\in\mathcal{A}^v_{\geqslant -m-1}$ for $m\neq 0$ and $\lambda_p\in\mathcal{A}^v_{\geqslant 0}$ for $m=0$, we have

$$[\lambda_p(T^*_{\lambda}\mathcal{A}^v_M)^c]^v_{\geqslant 0} = \lambda_p(T^*_{\lambda}\mathcal{A}^v_M)^c.$$

Hence, for arbitrary $f\in\mathcal{A}^v$:

$$f\circ(T^*_{\lambda}\mathcal{A}^v_M)^c = p^s[\lambda_p f]^v_{\geqslant 0}\,(T^*_{\lambda}\mathcal{A}^v_M)^c \subset (T^*_{\lambda}\mathcal{A}^v_M)^c$$

and thus$(T^*_{\lambda}\mathcal{A}^{\nu}_M)^c$ is an ideal without any further restrictions. Moreover,

$$\operatorname{Tr}_v(f\circ(T^*_{\lambda}\mathcal{A}^v_M)^c) = 0 \qquad\Longleftrightarrow\qquad m\geqslant -1,$$

which finishes the proof. $\square$

Lemma 6.5. *In each case when the metric (6.9) is nondegenerate on $\mathcal{A}^{\nu}_M$ (Proposition 6.3) there is a Frobenius algebra structure defined in the cotangent bundle $T^*\mathcal{A}^{\nu}_M$ by the respective quotient algebra $\mathcal{A}^{\nu}/(T^*_{\lambda}\mathcal{A}^{\nu}_M)^c$ (Proposition 6.4). The respective unit vector fields are given by*

$$e = \begin{cases}1-\lambda_p & \textit{for } \nu=\infty,\ s=0 \textit{ and } \deg_{\infty}\lambda = 1,\\ 1-\frac{1}{u_1}\lambda_p & \textit{for } \nu=0,\ s=1 \textit{ and } \deg_0\lambda=-1,\\ 1 & \textit{otherwise.}\end{cases} \tag{6.11}$$

So, almost always $e=1$. In all the above cases but one the respective unit vector fields are flat. The exception is the case of $\nu=0$, $s=1$ and $\deg_0\lambda=-1$.

Proof. In fact, the first part of the lemma is a corollary to Propositions 6.3 and 6.4. It is only left to show that in each case there exists unit element (1-form) for the respective multiplication (6.8). We will take advantage from the nondegeneracy of the metric (6.9) and we will use the canonical isomorphism (6.10). Notice that all the unit vector fields

(6.11) belong to the respective tangent spaces. Let ε be 1-form such that $\varepsilon^\sharp = e$. By relation (5.12) for arbitrary $\beta \in \Lambda^1(\mathcal{A}_M^\nu)$ we have

$$(\varepsilon \circ \beta)^\sharp = p^s e[\lambda_p \beta]^\nu_{\geqslant 0} - p^s \lambda_p [e\beta]^\nu_{\geqslant 0} = p^s \lambda_p [e\beta]^\nu_{<0} - p^s e[\lambda_p \beta]^\nu_{<0}\,.$$

In the case when $e = 1$ one finds immediately that $(\varepsilon \circ \beta)^\sharp = \beta^\sharp$. In the remaining cases it is slightly more involved to check. This shows that 1-forms ε are units in the respective quotient algebras.

The co-unity ε is closed on $\mathcal{A}_M^\nu$, which is consequence of Proposition 6.1 and Proposition 4.10. Therefore, for the flatness of e it is sufficient to show (2.5) or

$$\mathrm{Lie}_e\, \eta^* = 0 \qquad \iff \qquad \mathrm{Lie}_e \circ = 0,$$

where the equivalence follows from the definition (6.9) and the fact that $\mathrm{Lie}_e \mathrm{Tr} = 0$. To obtain $\mathrm{Lie}_e \circ$ one should use similar computation as in the proof of next Lemma 6.6. When $e = 1$ the computation of $\mathrm{Lie}_e \circ$ is straightforward, for $e = 1 - \lambda_p$ it is slightly more involved. In the case of $e = 1 - \frac{1}{u_1}\lambda_p$, which is the exception, the relation $\mathrm{Lie}_e \circ = 0$ does not hold. $\square$

Lemma 6.6. *For the Frobenius algebras from Lemma 6.5 the quasi-homogeneity relation,* $\mathrm{Lie}_E \circ = (d-1)\circ$, *in the case of $\nu = \infty$ hold for the Euler vector field and the weight given by*

$$E = \lambda - \frac{1}{n} p \lambda_p \qquad \text{and} \qquad d = 1 + (s-1)\frac{2}{n}. \tag{6.12}$$

In the renaming cases the Euler vector field has the form $E = \lambda$ with $d = 1$.

Proof. We will present the detailed proof only for the case of $\nu = \infty$. First notice that the vector field $E = \lambda - \frac{1}{n} p \lambda_p$ belongs to $T_\lambda \mathcal{A}_M^\infty$. To compute $\mathrm{Lie}_E \circ$ we will use the formula (A.5), that is

$$(\mathrm{Lie}_E \circ)(\alpha, \beta) = (\mathrm{D}_E \circ)(\alpha, \beta) + \mathrm{D}^*_{\alpha\circ\beta} E - \mathrm{D}^*_\alpha E \circ \beta - \alpha \circ \mathrm{D}^*_\beta E. \tag{6.13}$$

Let $X \in \mathfrak{X}(\mathcal{A}_M^\infty)$ and $\alpha \in \Lambda^1(\mathcal{A}_M^\infty)$, then $\mathrm{D}_X E = X - \frac{1}{n} p X_p$ and

$$\langle \mathrm{D}^*_\alpha E, X\rangle = \langle \alpha, \mathrm{D}_X E\rangle = \mathrm{Tr}_\infty(\alpha\, \mathrm{D}_X E) = \mathrm{Tr}_\infty\left(\left(\frac{n-s+1}{n}\alpha + \frac{1}{n} p \alpha_p\right) X\right),$$

where the integration by parts is used in the trace form (6.2) ($\nu = \infty$). Hence, $\mathrm{D}^*_\alpha E = \frac{n-s+1}{n}\alpha + \frac{1}{n} p\, \alpha_p$. The subsequent terms in (6.13) have the form:

$$\begin{aligned}
(\mathrm{D}_E \circ)(\alpha, \beta) &= \ell(p^s E_p \alpha)\, \beta + \alpha \ell(p^s E_p \beta) \\
&= \frac{n-1}{n} \alpha \circ \beta - \frac{1}{n} \ell\big(p^{s+1} \lambda_{2p} \alpha\big)^\infty_{\geqslant s} \beta - \frac{1}{n} \alpha \ell\big(p^{s+1} \lambda_{2p} \beta\big)^\infty_{\geqslant s}, \\
\mathrm{D}^*_{\alpha\circ\beta} E &= \frac{n-s+1}{n} \alpha \circ \beta + \frac{1}{n} p \left(\alpha \circ \beta\right)_p, \\
\mathrm{D}^*_\alpha E \circ \beta &= \ell(p^s \lambda_p\, \mathrm{D}^*_\alpha E)\, \beta + \mathrm{D}^*_\alpha E\, \ell(p^s \lambda_p \beta) \\
&= \frac{n-s+1}{n} \alpha \circ \beta + \frac{1}{n} \ell\big(p^{s+1} \lambda_p \alpha_p\big)^\infty_{\geqslant s} \beta + \frac{1}{n} p\, \alpha_p \ell(p^s \lambda_p \beta)^\infty_{\geqslant s}\,.
\end{aligned}$$

Substituting the above terms into (6.13) and using the fact that the relation $p\partial_p \ell(\cdot) = \ell(p\partial_p \cdot)$ holds for (6.3) (for $s = 0$ or 1) we obtain the following equality

$$\mathrm{Lie}_E \circ = 2\frac{s-1}{n} \circ .$$

For finite $\nu = 0$ or v we have $E = \lambda \in T_\lambda \mathcal{A}^\nu_M$ and $D_X E = X$, $D^*_\alpha E = \alpha$. The corresponding computation of the quasi-homogeneity relation is similar and adequately simpler then the above one. □

Combining the above lemmas and propositions with Theorem 5.4 we have the following result:

Theorem 6.7.

- *For $s = 0, 1$ there is a structure of Frobenius manifold on $\mathcal{A}^\infty_M$ if $\deg_\infty \lambda \geqslant 1$.*
- *There is a structure of Frobenius manifold on $\mathcal{A}^0_M$ for $s = 0$ if $\deg_0 \lambda = 0$ and for $s = 1$ if $\deg_0 \lambda \geqslant 1$.*
- *In the case of $s = 1$ and $\deg_0 \lambda = -1$ there is a structure of Frobenius manifold on $\mathcal{A}^0_M$ with nonflat unit vector field.*
- *For finite $\nu = v$ and $s = 0, 1$ there is a structure of Frobenius manifold on $\mathcal{A}^v_M$ if $\deg_v \lambda \geqslant -1$.*

Remark 6.8. Consider the second contravariant metric on $\mathcal{A}^\nu_M$ defined by (4.9) for $r = \ell^*$ and $E = \lambda$, which takes the form

$$g^*(\alpha, \beta) = \mathrm{Tr}_\nu(\lambda\, \alpha \circ \beta) \qquad \alpha, \beta \in \Lambda^1(\mathcal{A}^\nu_M). \tag{6.14}$$

This metric is well defined and nondegenerate at a generic point of manifold subspaces $\mathcal{A}^\nu_M$ for finite $\nu = 0$ or v and $s = 0, 1$. This can be showed in a similar way to the proof of Proposition 6.3. In these cases, since $E = \lambda$ is a Euler vector field, the metric (6.14) coincides with the intersection form (2.9). However, for $\nu = \infty$ the metric (6.14) is well defined and nondegenerate only on the spaces:

$$\text{for } s = 0: \quad \widetilde{\mathcal{A}}^\infty_M = \left\{ \lambda(p) = p^n + \widetilde{u} p^{n-1} + u_{n-2} p^{n-2} + \ldots \right\},$$
$$\text{for } s = 1: \quad \widetilde{\mathcal{A}}^\infty_M = \left\{ \lambda(p) = \widetilde{u} p^n + u_{n-1} p^{n-1} + u_{n-2} p^{n-2} + \ldots \right\}.$$

Hence, to obtain the second metric on $\mathcal{A}^\infty_M$ one must to carry out the reduction procedure with respect to the constrain $\widetilde{u} = 0$. After reduction one gets the metric in the form $g^*_{\mathrm{red}}(\alpha, \beta) = \langle \alpha^\sharp, \beta \rangle$, where

$$\alpha^\sharp = p^s \lambda [\lambda_p \alpha]^\nu_{\geqslant 0} - p^s \lambda_p [\lambda \alpha]^\nu_{\geqslant 0} + \frac{1}{n} p^s \lambda_p [\lambda_p \alpha]^\infty_{-1} \qquad s = 0, 1.$$

The above reduction is in fact equivalent to the so-called Dirac reduction of Lie-Poisson brackets considered in [34]. Using the relation $[\,\cdot\,]^\infty_{-1} = [p\,\cdot\,]^\infty_{\geqslant 0} - p[\,\cdot\,]^\infty_{\geqslant 0}$ one finds that the metric g^*_{red} coincides with the intersection form (2.9), that is

$$g^*_{\mathrm{red}}(\alpha, \beta) = \langle E, \alpha \circ \beta \rangle \equiv \mathrm{Tr}_\infty(E\, \alpha \circ \beta),$$

where $E = \lambda - \frac{1}{n} p \lambda_p$. The flatness of g^*_{red} and compatibility with η^* is a consequence of the fact that g^*_{red} is a intersection form. Notice that $E = \lambda - \frac{1}{n} p \lambda_p$ does not fulfil assumptions from Subsection 4.2 and the theorem cannot be used with this choice of E.

Remark 6.9. The particular infinite-dimensional Frobenius manifold $\mathcal{A}_M^\infty$ for $s=0$ and $n=1$ corresponds to the Frobenius manifold associated with dKP equation or Benney chain, which was constructed in [28]. On the other-hand in [9] there was formulated infinite-dimensional Frobenius manifold associated with two-component Toda chain, which corresponds to the following direct sum $\mathcal{A}_M^\infty \oplus \mathcal{A}_M^0$ for $s=1$ and $n=m=0$.

6.4. **Frobenius structure on the spaces of meromorphic functions.** Let us consider the algebra of meromorphic functions on the Riemann sphere with prescribed marked points, poles or zeros, at ∞, 0 and two families of finite (not fixed) points $a_1,\dots a_L$ and $v_1,\dots,v_K$ that can vary over the complex plane. Therefore, we define the algebra in the form

$$\mathcal{A} = \mathbb{C}[\![p, p^{-1}, (p-a_1)^{-1},\dots,(p-a_L)^{-1},(p-v_1)^{-1},\dots,(p-v_K)^{-1}]\!],$$

where $p\in\mathbb{C}^*$. The elements $p,(p-a_i),(p-v_j)$ and their inverses are the generators of the algebra $\mathcal{A}$ with obvious relations between them.

The underlying manifold subspaces $\mathcal{A}_M$ of $\mathcal{A}$ on which we are going to define the structure of Frobenius manifold are reductions of the infinite-dimensional Frobenius manifolds associated to the formal Laurent spaces considered above. Hence, we must take into consideration the constraints from Theorem 6.7 on the degrees of the meromorphic functions $\lambda(p)\in\mathcal{A}_M$.

Accordingly, from Theorem 6.7 it follows that for $s=0$ the meromorphic functions $\lambda(p)\in\mathcal{A}_M$ cannot have zero or pole at $p=0$, that is we must require $\deg_0\lambda=0$. For $s=1$ there must be singularity at $p=0$ of order $\deg_0\lambda\geqslant 1$ or zero of order one, that is $\deg_0\lambda=-1$. All non fixed zeros a_i of $\lambda(p)$ must be of order one, $\deg_{a_i}\lambda=1$, and all non fixed poles v_j must be of degree $\deg_{v_j}\lambda\geqslant 1$. Besides, the meromorphic functions $\lambda(p)$ must have singularity at infinity of order $\deg_\infty\lambda\geqslant 1$ and the normalisation from (6.7) must be taken into account.

Let $n:=\deg_\infty\lambda$, $m_0:=\deg_0\lambda$ and $m_j:=\deg_{v_j}\lambda$ be fixed. Consequently, we define the underlying manifold subspace of $\mathcal{A}$ by

$$\mathcal{A}_M = \left\{ \lambda(p) = \frac{\prod_{i=1}^L (p-a_i)}{p^{m_0}\prod_{i=1}^K (p-v_j)^{m_j}} \;\middle|\; a_i, v_i\in\mathbb{C}, \sum_{j=1}^L a_i = \sum_{j=1}^K m_j v_j \text{ for } s=0 \right\}, \tag{6.15}$$

where we require that $n = L-\sum_{j=0}^K m_j\geqslant 1$ and $m_j\geqslant 1$. Besides, for $s=0$ we must have $m_0=0$ and for $s=1$ we must have $m_0\geqslant 1$ or $m_0=-1$. The coefficients a_i and v_j constitute coordinates on the underlying manifold, which is of dimension $N:=K+L+s-1$.

The related tangent spaces are spanned by the derivation of λ with respect to the coordinates, that is

$$T_\lambda\mathcal{A}_M = \operatorname{span}\left\{\frac{\partial\lambda}{\partial a_1},\dots,\frac{\partial\lambda}{\partial a_l},\frac{\partial\lambda}{\partial v_1},\dots,\frac{\partial\lambda}{\partial v_k}\right\}.$$

Let $\Gamma:=\{\infty, 0\,(\text{if } s=1), v_1,\dots,v_K\}$ be the set that consists of poles of meromorphic functions $\lambda\in\mathcal{A}_M$. Define $\widetilde{\Gamma}:=\Gamma\setminus\{\infty\}$. We can establish, the related dual spaces with respect to the trace forms (6.2) for $\nu\in\Gamma$ as appropriate subsets of algebras of Laurent

series, that is $T^*_\lambda\mathcal{A}^\nu_M \subset \mathcal{A}^\nu$. They form is not unique, more precisely, they are given by appropriate quotient spaces, see the following lemma. Possible and useful representations are:

$$T^*_\lambda\mathcal{A}^\infty_M = \left\{\gamma_1 p^{N-n} + \ldots + \gamma_N p^{1-n} \mid \gamma_1, \ldots, \gamma_N \in \mathbb{C}\right\} \subset \mathcal{A}^\infty_{\geqslant 1-n} \tag{6.16}$$

for $\nu = \infty$ and

$$T^*_\lambda\mathcal{A}^\nu_M = \left\{\gamma_1 (p-\nu)^m + \ldots + \gamma_N (p-\nu)^{m+1-N} \mid \gamma_1, \ldots, \gamma_N \in \mathbb{C}\right\} \subset \mathcal{A}^\nu_{\leqslant m} \tag{6.17}$$

for $\nu \in \widetilde{\Gamma}$, where $m = \deg_\nu \lambda$.

For each $\mathcal{A}_M$ the duality paring, such as (4.2), can be defined by the trace form (6.2) for different $\nu \in \Gamma$. As result, a 1-form γ on the underlying manifold can have different representations $\gamma_\nu \in T^*_\lambda\mathcal{A}^\nu_M$, such that at point $\lambda \in \mathcal{A}_M$:

$$\langle \gamma, X\rangle_\lambda \equiv \operatorname{Tr}_\infty(X\gamma_\infty) = \operatorname{Tr}_\nu(X\gamma_\nu) \qquad \nu \in \widetilde{\Gamma}, \tag{6.18}$$

where $X \in T_\lambda\mathcal{A}_M$ is arbitrary. Then, for each $\nu \in \Gamma$ we can define the related contravariant metric on $\mathcal{A}_M$ (6.9) and the related multiplication in the cotangent bundle $T^*\mathcal{A}^\nu_M$ using (6.8). We will show that these structures defined for different $\nu \in \Gamma$ are isomorphic.

Lemma 6.10. *Each metric (6.9), defined for $\nu \in \Gamma$, is nondegenerate at a generic point $\lambda \in \mathcal{A}_M$. Moreover, in each case the respective metric (6.9) is compatible with the structure of quotient algebra $\mathcal{A}^\nu/(T^*_\lambda\mathcal{A}_M)^c$, where $(T^*_\lambda\mathcal{A}_M)^c \equiv \mathcal{A}\setminus T^*_\lambda\mathcal{A}^\nu_M$.*

Proof. We will consider in detail only the case of $\nu = \infty$. Let $\mathbb{C}_r[\![p]\!]$ denote the space of polynomials in p of order at most r and define

$$factor := p^{m_0}\prod_{i=1}^K (p - v_j)^{m_j+1}.$$

All the following computations are made at a generic point $\lambda \in \mathcal{A}_M$. One can see that $T_\lambda\mathcal{A}_M \cong factor^{-1} \times \mathbb{C}_{N-1}[\![p]\!]$. Hence, we can define the related cotangent space with respect to the trace form (6.2) with $\nu = \infty$ as $T^*_\lambda\mathcal{A}^\infty_M = factor \times p^{N-s} \times \mathbb{C}_{N-1}[\![p]\!]$.

For arbitrary $\alpha \in T^*_\lambda\mathcal{A}^\infty_M$ one finds that

$$\begin{aligned} factor \times p^s[\lambda_p\alpha]^\nu_{\geqslant 0} &\in \mathcal{A}^\infty_{\geqslant m_0+s} \\ factor \times p^s\lambda_p[\alpha]^\nu_{\geqslant 0} &\in \mathcal{A}^\infty_{\geqslant 0} \end{aligned} \quad\Longrightarrow\quad \alpha^\sharp \in factor^{-1} \times \mathcal{A}^\infty_{\geqslant 0},$$

where (6.10) is used. On the other hand

$$\begin{aligned} factor \times p^s\lambda_p[\alpha]^\nu_{<0} &\in \mathcal{A}^\infty_{\leqslant N-1} \\ factor \times p^s[\lambda_p\alpha]^\nu_{<0} &\in \mathcal{A}^\infty_{\leqslant N-n} \end{aligned} \quad\Longrightarrow\quad \alpha^\sharp \in factor^{-1} \times \mathcal{A}^\infty_{\leqslant N-1}.$$

Hence, one can conclude that $\operatorname{Im}\sharp = factor^{-1} \times \mathbb{C}_{N-1}[\![p]\!]$, that is the image of $\sharp$ spans $T_\lambda\mathcal{A}_M$.

The complement of $T^*_\lambda\mathcal{A}^\infty_M$ to $\mathcal{A}^\infty$ is given by $(T^*_\lambda\mathcal{A}^\infty_M)^c = factor \times \left(\mathcal{A}^\infty_{\geqslant s} \oplus \mathcal{A}^\infty_{<s-N}\right)$. For arbitrary $g \in factor \times \mathcal{A}^\infty_{\geqslant s}$ and $h \in factor \times \mathcal{A}^\infty_{<s-N}$ we see that $\lambda_p g \in \mathcal{A}^\infty_{\geqslant 0}$ and $\lambda_p h \in \mathcal{A}^\infty_{<0}$. Hence for arbitrary $f \in \mathcal{A}$:

$$f \circ g = p^s[\lambda_p f]^\nu_{\geqslant 0}\, g \in factor \times \mathcal{A}^\infty_{\geqslant 2s} \subset (T^*_\lambda\mathcal{A}_M)^c$$

and

$$f \circ h = -p^s [\lambda_p f]^{\nu}_{<0}\, h \in factor \times \mathcal{A}^{\infty}_{<2s-N-1} \subset (T^*_\lambda \mathcal{A}_M)^c,$$

where the multiplication is defined by (6.8) with $\nu = \infty$. Thus $(T^*_\lambda \mathcal{A}_M)^c$ is an ideal.

Now, the compatibility of the quotient structure with the metric (6.9) follows from the relations:

$$\mathrm{Tr}_\infty\big(factor \times \mathcal{A}^{\infty}_{\geqslant 2s}\big) = -\operatorname{res}_{p=\infty} \mathcal{A}^{\infty}_{\geqslant m_0+s} = 0$$

and

$$\mathrm{Tr}_\infty\big(factor \times \mathcal{A}^{\infty}_{<2s-N-1}\big) = -\operatorname{res}_{p=\infty} \mathcal{A}^{\infty}_{<-n-1} = 0.$$

The remaining cases of $\nu \in \widetilde{\Gamma}$ can be proven in a similar fashion or can be based on the next theorem. □

Let 1-form α be represented by $\alpha_\nu \in T^*_\lambda \mathcal{A}^\nu_M$, then let $\alpha^\sharp_\nu \in T_\lambda \mathcal{A}_M$ be given by (6.10) and let $\alpha_\nu \circ \beta_\nu$ means the product of two 1-forms in the tangent bundle $T^*\mathcal{A}^\nu_M$ defined by the multiplication (6.8).

Theorem 6.11. *The metrics defined by* (6.10) *for different $\nu \in \Gamma$ on (fixed) $\mathcal{A}_M$ are equivalent, that is the following relation is true:*

$$\eta^*(\alpha,\beta) := \mathrm{Tr}_\infty\big(\alpha^\sharp_\infty \beta_\infty\big) \equiv \mathrm{Tr}_\nu\big(\alpha^\sharp_\nu \beta_\nu\big) \qquad \nu \in \widetilde{\Gamma}.$$

This means that for arbitrary 1-form α the following equality is also valid

$$\alpha^\sharp := \alpha^\sharp_\infty \equiv \alpha^\sharp_\nu \in T_\lambda \mathcal{A}_M \qquad \nu \in \widetilde{\Gamma}. \tag{6.19}$$

Similarly, the multiplications in the cotangent bundles $T^*\mathcal{A}^\nu_M$ defined by (6.8) *for different $\nu \in \Gamma$ are isomorphic.*

Proof. Let $\nu \in \widetilde{\Gamma}$. Using (6.18) and the residue theorem one finds that

$$\begin{aligned}
\mathrm{Tr}_\nu\big(\alpha^\sharp_\nu \beta_\nu\big) &= \mathrm{Tr}_\infty\big(\alpha^\sharp_\nu \beta_\infty\big) = -\operatorname{res}_{p=\infty}\big([\lambda_p \alpha_\nu]^\nu_{\geqslant 0}\, \beta_\infty - \lambda_p [\alpha_\nu]^\nu_{\geqslant 0}\, \beta_\infty\big)\\
&= -\operatorname{res}_{p=\infty}\big([\lambda_p \alpha_\nu]^\nu_{\geqslant 0} [\beta_\infty]^\infty_{<0} - [\alpha_\nu]^\nu_{\geqslant 0} [\lambda_p \beta_\infty]^\infty_{<0}\big)\\
&= \operatorname{res}_{p=\nu}\big([\lambda_p \alpha_\nu]^\nu_{\geqslant 0} [\beta_\infty]^\infty_{<0} - [\alpha_\nu]^\nu_{\geqslant 0} [\lambda_p \beta_\infty]^\infty_{<0}\big)\\
&= \operatorname{res}_{p=\nu}\big(\lambda_p \alpha_\nu [\beta_\infty]^\infty_{<0} - \alpha_\nu [\lambda_p \beta_\infty]^\infty_{<0}\big) = \mathrm{Tr}_\nu\big(\beta^\sharp_\infty \alpha_\nu\big) = \mathrm{Tr}_\infty\big(\beta^\sharp_\infty \alpha_\infty\big)\,.
\end{aligned}$$

Hence, the equivalence of metrics (6.9) is proven.

The multiplications in the cotangent bundles $T^*\mathcal{A}^\nu_M$ defined by (6.8) are mutually isomorphic if for arbitrary $X \in T_\lambda \mathcal{A}_M$ the following relation holds:

$$\langle X, \beta \circ \gamma\rangle = \mathrm{Tr}_\infty(X(\beta_\infty \circ \gamma_\infty)) \equiv \mathrm{Tr}_\nu(X(\beta_\nu \circ \gamma_\nu))\,.$$

Using (6.19) we can take $X = \alpha^\sharp_\infty \equiv \alpha^\sharp_\nu$, where the 1-form α is arbitrary. Hence, the above relation is equivalent to

$$\mathrm{Tr}_\infty(\alpha_\infty \circ \beta_\infty \circ \gamma_\infty) = \mathrm{Tr}_\nu(\alpha_\nu \circ \beta_\nu \circ \gamma_\nu)\,,$$

where α, β, γ are arbitrary 1-forms. Now, by (5.12) for $\nu \in \Gamma$:

$$\begin{aligned}(\alpha_\nu \circ \beta_\nu)^\sharp &= p^s \alpha^\sharp [\lambda_p \beta_\nu]^\nu_{\geqslant 0} - p^s \lambda_p \left[\alpha^\sharp \beta_\nu\right]^\nu_{\geqslant 0} \\ &= p^s \lambda_p \left[\alpha^\sharp \beta_\nu\right]^\nu_{<0} - p^s \alpha^\sharp [\lambda_p \beta_\nu]^\nu_{<0} .\end{aligned}$$

Hence, in the same manner as before, using (6.18) and the residue theorem, one finds that

$$\begin{aligned}\mathrm{Tr}_\nu(\alpha_\nu \circ \beta_\nu \circ \gamma_\nu) &= \mathrm{Tr}_\nu\Big(\alpha_\nu (\beta_\nu \circ \gamma_\nu)^\sharp\Big) = \mathrm{Tr}_\infty\Big(\alpha_\infty (\beta_\nu \circ \gamma_\nu)^\sharp\Big) \\ &= -\operatorname{res}_{p=\infty}\Big(\alpha_\infty \beta^\sharp [\lambda_p \gamma_\nu]^\nu_{\geqslant 0} - \lambda_p \alpha_\infty \left[\beta^\sharp \gamma_\nu\right]^\nu_{\geqslant 0}\Big) \\ &= \operatorname{res}_{p=\nu}\Big(\left[\alpha_\infty \beta^\sharp\right]^\infty_{<0} \lambda_p \gamma_\nu - [\lambda_p \alpha_\infty]^\infty_{<0} \beta^\sharp \gamma_\nu\Big) \\ &= \mathrm{Tr}_\nu\Big((\alpha_\infty \circ \beta_\infty)^\sharp \gamma_\nu\Big) = \mathrm{Tr}_\infty\Big((\alpha_\infty \circ \beta_\infty)^\sharp \gamma_\infty\Big) = \mathrm{Tr}_\infty(\alpha_\infty \circ \beta_\infty \circ \gamma_\infty) .\end{aligned}$$

This finishes the proof. □

Lemma 6.12. *On each underlying manifold $\mathcal{A}_M$ there is a structure of Frobenius algebra defined by the quotient algebra $\mathcal{A}_M/(T^*_\lambda \mathcal{A}^\nu_M)^c \cong T^*_\lambda \mathcal{A}^\nu_M$ and the respective metric (6.9). These structures are equivalent for different $\nu \in \Gamma$. The related unit vector fields are given by*

$$e = \begin{cases} 1 - \lambda_p & \text{for } \nu = \infty,\ s = 0 \text{ and } \deg_\infty \lambda = 1, \\ 1 - \frac{1}{u_1}\lambda_p & \text{for } \nu = 0,\ s = 1 \text{ and } \deg_0 \lambda = -1, \\ 1 & \text{otherwise.} \end{cases}$$

So, almost always $e = 1$. In all the above cases but one the respective unit vector fields are flat. The exception is the case of $s = 1$ with $\deg_0 \lambda = -1$.

For the above Frobenius algebras the quasi-homogeneity relation, $\mathrm{Lie}_E \circ = (d-1)\circ$ is fulfilled by the Euler vector field $E = \lambda - \frac{1}{n} p \lambda_p$ and the weight $d = 1 + (s-1)\frac{2}{n}$.

The first part of Lemma (6.12) is a corollary to previous propositions and the proof of the second part is practically the same as the respective parts of the proofs of Lemma 6.5 and Lemma 6.6.

Proposition 6.13. *The flat coordinates for the contravariant metric (6.9) defined on $\mathcal{A}_M$ are given by*

$$t^i_\infty := \frac{1}{1 - \frac{i}{n}} \mathrm{Tr}_\infty \lambda^{1-\frac{i}{n}} \quad \text{for} \quad 1 \leqslant i \leqslant n-1$$

and for $1 - s \leqslant k \leqslant K$ by [3]

$$t^j_{v_k} = \begin{cases} \frac{1}{1-\frac{j}{m_k}} \mathrm{Tr}_{v_k} \lambda^{1-\frac{j}{m_k}} & \text{for} \quad 0 \leqslant j < m_k, \\ \mathrm{Tr}_{v_k} \log \lambda + \frac{m_k}{n} \mathrm{Tr}_\infty \log \lambda & \text{for} \quad j = m_k. \end{cases} \tag{6.20}$$

The respective differentials are

$$dt^i_\infty := \left[\lambda^{-\frac{i}{n}}\right]^\infty_{\geqslant 1-n} \in T^*_\lambda \mathcal{A}^\infty_M \qquad 1 \leqslant i \leqslant n-1$$

[3] Care must be taken when calculating the traces of terms involving logarithmic singularities, see Appendix D.

and

$$dt^j_{v_k} := \left[\lambda^{-\frac{j}{m_k}}\right]^{v_k}_{\leqslant m_k} \in T^*_\lambda \mathcal{A}^{v_k}_M \qquad 1-s \leqslant k \leqslant K, \quad 0 \leqslant j \leqslant m_k.$$

Proof. First we will show that dt^i_ν are flat 1-forms with respect to the metric defined by (6.9) for $\nu \in \Gamma$. See Remark 4.8, it is sufficient to check if the equalities (4.18) (with $E = 1$) hold. Remind that $r = \ell^*$, where ℓ^* is given by (6.4).

For $\nu = \infty$ we take $\gamma^i_\infty = \lambda^{-\frac{i}{n}}$, for which $\deg_\infty \gamma^i_\infty = -i$. Hence, the conditions (4.18) are satisfied since $[\gamma^i_\infty]^\infty_{\geqslant 0} = 0$ for $i > 0$. Projecting γ^i_∞ on the cotangent space (6.16), one finds that the only nonzero projections are for $1 \leqslant i \leqslant n-1$. Similarly for $\nu \in \widetilde{\Gamma}$ we take $\gamma^j_\nu = \lambda^{-\frac{i}{m}}$, where $m = \deg_\nu \lambda$. Thus $\deg_\nu \gamma^j_\nu = -j$ and (4.18) hold as $[\gamma^j_\nu]^\nu_{<0} = 0$ for $j \geqslant 0$. The only nonzero projections of γ^j_ν on the respective cotangent space (6.17) are for $0 \leqslant j \leqslant m$.

The related flat coordinates are the respective (locally defined) functions t^i_ν so that

$$\gamma^i_\nu = dt^i_\nu \qquad \Longleftrightarrow \qquad \mathrm{D}_X t^i_\nu = \mathrm{Tr}_\nu\left(X\gamma^i_\nu\right),$$

where X is an arbitrary vector field on $\mathcal{A}_M$. □

Proposition 6.14. *The contravariant metric η^* on $\mathcal{A}_M$, defined by means of* (6.9)*, decomposes in flat coordinates from Proposition 6.13 into anti-diagonal blocks such that*

$$\eta^*\left(dt^i_\nu, dt^j_\nu\right) = m\,\delta_{i,m-j} \qquad m = \deg_\nu \lambda \qquad \nu \in \Gamma.$$

Proof. First notice that $(dt^i_\infty)^\sharp = p^s[\lambda_p dt^i_\infty]^\infty_{\geqslant 0}$ and $(dt^j_\nu)^\sharp = -p^s[\lambda_p dt^j_\nu]^\nu_{<0}$ for $\nu \in \widetilde{\Gamma}$. Then, we have

$$\begin{aligned}
\eta^*\left(dt^i_\infty, dt^j_\nu\right) &= \mathrm{Tr}_\infty\left(dt^i_\infty \left(dt^j_\nu\right)^\sharp\right) = -\operatorname{res}_{p=\infty}\left(\lambda^{-\frac{i}{n}}\left[\lambda_p \lambda^{-\frac{j}{m}}\right]^\nu_{<0}\right)\\
&= -\operatorname{res}_{p=\infty}\left(\left[\lambda^{-\frac{i}{n}}\right]^\infty_{\geqslant 0}\left[\lambda_p \lambda^{-\frac{j}{m}}\right]^\nu_{<0}\right) = 0
\end{aligned}$$

and for different $\nu, \nu' \in \widetilde{\Gamma}$:

$$\begin{aligned}
\eta^*\left(dt^i_\nu, dt^j_{\nu'}\right) &= \mathrm{Tr}_\nu\left(dt^i_\nu \left(dt^j_{\nu'}\right)^\sharp\right) = \operatorname{res}_{p=\nu}\left(\lambda^{-\frac{i}{m}}\left[\lambda_p \lambda^{-\frac{j}{m'}}\right]^{\nu'}_{<0}\right)\\
&= \operatorname{res}_{p=\nu}\left(\left[\lambda^{-\frac{i}{m}}\right]^\nu_{<0}\left[\lambda_p \lambda^{-\frac{j}{m'}}\right]^{\nu'}_{<0}\right) = 0.
\end{aligned}$$

Besides,

$$\begin{aligned}
\eta^*\left(dt^i_\infty, dt^j_\infty\right) &= \mathrm{Tr}_\infty\left(dt^i_\infty \left(dt^j_\infty\right)^\sharp\right) = -\operatorname{res}_{p=\infty}\left(\lambda^{-\frac{i}{n}}\left[\lambda_p \lambda^{-\frac{j}{n}}\right]^\infty_{\geqslant 0}\right)\\
&= -\operatorname{res}_{p=\infty}\left(\lambda_p \lambda^{-\frac{i+j}{n}}\right) = -n \operatorname{res}_{\lambda=\infty}\left(\lambda^{-\frac{i+j}{n}}\right) = n\,\delta_{i,n-j}
\end{aligned}$$

where in the residue integral one makes the change of coordinates $p \mapsto \lambda = p^n + \ldots$, taking into account the multiplicity of this transformation, $n = \deg_\infty \lambda$. Similarly, for $\nu \in \widetilde{\Gamma}$:

$$\begin{aligned}
\eta^*\left(dt^i_\nu, dt^j_\nu\right) &= \mathrm{Tr}_\nu\left(dt^i_\nu \left(dt^j_\nu\right)^\sharp\right) = \operatorname{res}_{p=\nu}\left(\lambda^{-\frac{i}{m}}\left[\lambda_p \lambda^{-\frac{j}{m}}\right]^\nu_{<0}\right)\\
&= \operatorname{res}_{p=\nu}\left(\lambda_p \lambda^{-\frac{i+j}{m}}\right) = -m \operatorname{res}_{\lambda=\infty}\left(\lambda^{-\frac{i+j}{m}}\right) = m\,\delta_{i,m-j},
\end{aligned}$$

where the change of coordinates is of the form $p \mapsto \lambda = \ldots + (p-\nu)^{-m}$ and $m = \deg_\nu \lambda$. □

Proposition 6.15. *The coefficients of the Euler vector field $E = \lambda - \frac{1}{n} p\lambda_p$ in the flat coordinates, that is $E^i_\nu \equiv \langle dt^i_\nu, E\rangle$, for $\nu = \infty$ are*

$$E^i_\infty = \left(\frac{1-s}{n} + \frac{n-i}{n}\right) t^i_\infty \quad \textit{for} \quad 1 \leqslant i \leqslant n-1$$

and for $\nu \in \widetilde{\Gamma}$ are

$$E^j_\nu = \begin{cases} \left(\frac{1-s}{n} + \frac{m-j}{m}\right) t^j_\nu & \textit{for} \quad 0 \leqslant j < m, \\ \frac{1}{n} t^m_\nu & \textit{for} \quad s = 0,\ j = m \\ \frac{m_0}{n} + 1 & \textit{for} \quad s = 1,\ \nu = 0,\ j = m \\ \frac{m}{n} & \textit{for} \quad s = 1,\ \nu \neq 0,\ j = m, \end{cases}$$

where $m = \deg_\nu \lambda$.

Proof. For $\nu = \infty$ the derivation is as follows

$$\begin{aligned} E^i_\infty &= \mathrm{Der}_E\, t^i_\infty = \mathrm{Tr}_\infty\left(\lambda^{-\frac{i}{n}} E\right) = -\operatorname{res}_{p=\infty}\left(p^{-s}\lambda^{-\frac{i}{n}}\left(\lambda - \frac{1}{n}p\lambda_p\right)\right) \\ &= -\left(1 + \frac{1-s}{n-i}\right)\operatorname{res}_{p=\infty} \lambda^{1-\frac{i}{n}} = \left(\frac{1-s}{n} + \frac{n-i}{n}\right) t^i_\infty. \end{aligned}$$

In particular for $s = 1$, $\nu = 0$ and $j = m_0$ we have

$$\begin{aligned} \langle dt^{m_0}_0, E\rangle &= \mathrm{Der}_E\, t^{m_0}_0 = \mathrm{Tr}_0\left(\lambda^{-1}E\right) + \frac{m_0}{n}\mathrm{Tr}_\infty\left(\lambda^{-1}E\right) \\ &= \operatorname{res}_{p=0}\left(p^{-1} - \frac{1}{n}\lambda^{-1}\lambda_p\right) - \frac{m_0}{n}\operatorname{res}_{p=\infty}\left(p^{-1} - \frac{1}{n}\lambda^{-1}\lambda_p\right) \\ &= \operatorname{res}_{p=0}\left(p^{-1} + \frac{m_0}{n}p^{-1}\right) - \frac{m_0}{n}\operatorname{res}_{p=\infty}\left(p^{-1} - p^{-1}\right) = \frac{m_0}{n} + 1. \end{aligned}$$

The computation in all other cases is similar. □

In order to use Lemma 2.4 we need the following proposition, proof of which is straightforward.

Proposition 6.16. *Define, by means of the flat coordinates from Proposition 6.13, functions $\mathcal{H}^{\nu,i}_{(0)} := t^i_\nu$. The recurrence formula (5.11) takes the form*

$$\frac{\partial \mathcal{H}^{\nu,i}_{(n)}}{\partial \lambda} = \mathcal{H}^{\nu,i}_{(n-1)}.$$

Hence, we have

$$\mathcal{H}^{\infty,i}_{(1)} := \frac{1}{1-\frac{i}{n}}\frac{1}{2-\frac{i}{n}}\mathrm{Tr}_\infty \lambda^{2-\frac{i}{n}} \quad \textit{for} \quad 1 \leqslant i \leqslant n-1$$

and for $1 - s \leqslant k \leqslant K$ we have

$$\mathcal{H}^{v_k,j}_{(1)} = \begin{cases} \frac{1}{1-\frac{j}{m_k}}\frac{1}{2-\frac{j}{m_k}}\mathrm{Tr}_{v_k}\lambda^{2-\frac{j}{m_k}} & \textit{for} \quad 0 \leqslant j < m_k, \\ \mathrm{Tr}_{v_k}(\lambda\log\lambda - \lambda) + \frac{m_k}{n}\mathrm{Tr}_\infty(\lambda\log\lambda - \lambda) & \textit{for} \quad j = m_k. \end{cases} \tag{6.21}$$

Now, combining the above results and using Lemma 2.4 we have the following theorem.

Theorem 6.17. *In all cases but one there is a structure of Frobenius manifold on $\mathcal{A}_M$. The exception is the case of $s = 1$ with $\deg_0 \lambda = -1$, when there is a structure of Frobenius manifold with a nonflat unit vector field. All the ingredients of these structures are defined above. The respective prepotential functions have the following form*

$$\mathcal{F} = \frac{1}{3-d}\left(\frac{1}{n}\sum_{i=1}^{n-1} E_\infty^i \mathcal{H}_{(1)}^{\infty,n-i} + \sum_{k=1-s}^{K}\frac{1}{m_k}\sum_{j=0}^{m_k} E_{v_k}^j \mathcal{H}_{(1)}^{v_k,m_k-j}\right), \tag{6.22}$$

where $d = 1 + (s-1)\frac{2}{n}$.

Remark 6.18. The class of Frobenius manifolds constructed in this section corresponds to the Frobenius manifolds classified in [14] and associated with Hurwitz spaces of zero genus. With respect to the related Landau-Ginzburg formalism, the cases of $s = 0$ and $s = 1$ correspond to the choice of a primary differential as $d\omega = dp$ and $d\omega = \frac{dp}{p}$, respectively. The particular case of $\mathcal{A}_M$ for $s = 0$, which consists of polynomial functions (6.23), is associated with Frobenius manifolds arising in the Saito's (singularity) theory labeled by A_{n-1}, see [14]. On the other-hand, $\mathcal{A}_M$ for $s = 1$, which consists of meromorphic functions with poles only at infinity and zero, is related with a class of Frobenius manifolds studied in [16] (see also [8]) associated with the extended affine Weyl groups of the A series. The explicit form of the prepotential (6.22) is a close analog of respective formulae given in [14] and [25] (for $s = 0$).

Now, we will illustrate the presented theory with few characteristic examples, some of them will contain more details of the related computations than other. The scheme is as follows. First, one needs to establish the manifold subspace $\mathcal{A}_M$ (6.15) and compute the flat coordinates $t_1, \ldots, t_N$ according to Proposition 6.13. The flat coordinates can be chosen so that the unit $e = \frac{\partial}{\partial t_1}$ (except the case of $s = 1$ with $m_0 = \deg_0 \lambda = -1$). Next, the prepotential function $\mathcal{F}$ is given by (6.22), where the formulae from Propositions 6.15 and 6.16 must be used. The respective Euler vector fields can be obtained from Lemma 6.12 or Proposition 6.15. Having the prepotential $\mathcal{F}$ coefficients of the covariant metric η and the structure constants of the multiplication in the tangent bundle can be easily computed from (2.6).

Example 6.19. Consider the manifold space $\mathcal{A}_M$, in the case of $s = 0$, that consists of meromorphic functions with only one pole at infinity of fixed order n:

$$\lambda = p^n + u_{n-2}p^{n-2} + \ldots + u_1 p + u_0 \qquad n \geqslant 2. \tag{6.23}$$

In this case the prepotential (6.22) takes the following 'symmetric' form:

$$\mathcal{F} = \frac{n^2}{2(n+1)}\sum_{i=1}^{n-1}\frac{n+1-i}{i(n^2-i^2)} \operatorname{res}_{p=\infty} \lambda^{\frac{n-i}{n}} \operatorname{res}_{p=\infty} \lambda^{\frac{n+i}{n}}.$$

In particular case of $n = 4$ we have

$$\lambda = p^4 + up^2 + vp + w \equiv p^4 + t_3p^2 + t_2p + t_1 + \frac{1}{8}t_3^2,$$

where the flat coordinates are given by

$$
\begin{aligned}
t_1 &\equiv t^1_\infty = -\frac{4}{3}\operatorname{res}_{p=\infty}\lambda^{\frac{3}{4}} = w - \frac{1}{8}u^2\\
t_2 &\equiv t^2_\infty = -2\operatorname{res}_{p=\infty}\lambda^{\frac{1}{2}} = v\\
t_3 &\equiv t^3_\infty = -4\operatorname{res}_{p=\infty}\lambda^{\frac{1}{4}} = u.
\end{aligned}
$$

Using the above formula one obtains

$$
\mathcal{F} = \frac{1}{8}t_1t_2^2 + \frac{1}{8}t_1^2t_3 - \frac{1}{64}t_2^2t_3^2 + \frac{t_3^5}{3840}.
$$

The related Euler vector field is

$$
E = t_1\frac{\partial}{\partial t_1} + \frac{3}{4}t_2\frac{\partial}{\partial t_2} + \frac{1}{2}t_3\frac{\partial}{\partial t_3} \qquad\Longleftrightarrow\qquad E(\lambda) = \lambda - \frac{1}{4}p\lambda_p
$$

and the weight $d = \frac{1}{2}$. The unit vector field is $e = \frac{\partial}{\partial t_1}$, since $e(\lambda) = 1$.

Example 6.20. Consider $\mathcal{A}_M$, for $s = 0$, that consists of functions with pole at infinity and two non-fixed 'finite' poles $v_1 = v$ and $v_2 = w$, all of order one, that is $n = m_1 = m_2 = 1$. Hence, $\lambda \in \mathcal{A}_M$ has the form

$$
\lambda = p + \frac{a}{p-v} + \frac{b}{p-w} = p + \frac{t_3}{p - t_1 - t_2} + \frac{t_4}{p - t_1 + t_2}.
$$

We take the following flat coordinates:

$$
\begin{aligned}
t_1 &\equiv \frac{1}{2}\left(t^1_v + t^1_w\right) = \frac{1}{2}(v+w)\\
t_2 &\equiv \frac{1}{2}\left(t^1_v - t^1_w\right) = \frac{1}{2}(v-w)\\
t_3 &\equiv t^0_v = a\\
t_4 &\equiv t^0_w = b,
\end{aligned}
$$

where, for $\nu = v, w$, by Proposition 6.13:

$$
t^0_\nu = \operatorname{res}_{p=\nu}\lambda, \qquad t^1_\nu = \operatorname{res}_{p=\nu}\log\lambda - \operatorname{res}_{p=\infty}\log\lambda.
$$

By Proposition 6.16:

$$
\begin{aligned}
\mathcal{H}^{\nu,0}_{(1)} &= \frac{1}{2}\operatorname{res}_{p=\nu}\lambda^2\\
\mathcal{H}^{\nu,1}_{(1)} &= \operatorname{res}_{p=\nu}(\lambda\log\lambda - \lambda) - \operatorname{res}_{p=\infty}(\lambda\log\lambda - \lambda)\,,
\end{aligned}
$$

hence using the formula (6.22) the prepotential is

$$
\mathcal{F} = t_1t_2(t_3 - t_4) + \frac{1}{4}(t_1^2 + t_2^2)(t_3 + t_4) + \frac{1}{2}t_3^2\log t_3 + t_3t_4\log t_2 + \frac{1}{2}t_4^2\log t_4.
$$

The related Euler vector field

$$
E = t_1\frac{\partial}{\partial t_1} + t_2\frac{\partial}{\partial t_2} + 2t_3\frac{\partial}{\partial t_3} + 2t_4\frac{\partial}{\partial t_4} \qquad\Longleftrightarrow\qquad E(\lambda) = \lambda - p\lambda_p,
$$

weight $d = -1$ and the unit vector field $e = \frac{\partial}{\partial t_1}$ as $e(\lambda) = 1 - \lambda_p$.

Example 6.21. Consider $\mathcal{A}_M$, for $s=1$, consisting of functions with poles at infinity, 0 and non-fixed 'finite' pole $v_1 = w$, all of order one, that is $n = m_0 = m_1 = 1$:

$$\lambda = p + u + \frac{v}{p} + \frac{s}{p-w} = p + t_1 + t_3 + \frac{e^{t_2}}{p} - \frac{t_3\, e^{t_4}}{p+e^{t_4}}.$$

The flat coordinates are:

$$t_1 \equiv t_0^0 = u - \frac{s}{w}, \qquad t_2 \equiv t_0^1 = \log v, \qquad t_3 \equiv t_w^0 = \frac{s}{w}, \qquad t_4 \equiv t_w^1 = \log(-w).$$

One obtains the prepotential

$$\mathcal{F} = \frac{1}{2}t_1^2 t_2 + t_1 t_3 t_4 + \frac{1}{2}t_3^2 t_4 + e^{t_2} + t_3 e^{t_2 - t_4} - t_3 e^{t_4} + \frac{1}{2}t_3^2 \log t_3.$$

Besides,

$$E = t_1 \frac{\partial}{\partial t_1} + 2\frac{\partial}{\partial t_2} + t_3 \frac{\partial}{\partial t_3} + \frac{\partial}{\partial t_4} \qquad \Longleftrightarrow \qquad E(\lambda) = \lambda - p\lambda_p,$$

$d = 1$ and $e = \frac{\partial}{\partial t_1}$ as $e(\lambda) = 1$.

Example 6.22. Consider $\mathcal{A}_M$, for $s=1$, consisting of functions with poles of order one at infinity and 0, that is $n = m_0 = 1$:

$$\lambda = p + u + \frac{v}{p} = p + t_1 + \frac{e^{t_2}}{p},$$

where the flat coordinates are $t_1 \equiv t_0^0 = u$ and $t_2 \equiv t_0^1 = \log v$. The potential is

$$\mathcal{F} = \frac{1}{2}t_1^2 t_2 + e^{t_2}.$$

Besides,

$$E = t_1 \frac{\partial}{\partial t_1} + 2\frac{\partial}{\partial t_2} \qquad \Longleftrightarrow \qquad E(\lambda) = \lambda - p\lambda_p,$$

$d = 1$ and $e = \frac{\partial}{\partial t_1}$ as $e(\lambda) = 1$. This is a celebrated example of Frobenius manifold corresponding to the quantum cohomology of complex projective line $\mathbb{P}^1$.

Example 6.23. Let $s = 1$ and $\mathcal{A}_M$ consists of functions with poles of order one at infinity and $v_1 = v$, that is $n = m_1 = 1$ as well as $m_0 = \deg_0 \lambda = -1$:

$$\lambda = p + u + \frac{uw}{p-w} = p + t_1 - \frac{t_1\, e^{t_2}}{p + e^{t_2}},$$

with flat coordinates $t_1 \equiv t_w^0 = u$ and $t_2 \equiv t_w^1 = \log(-w)$. The prepotential is

$$\mathcal{F} = \frac{1}{2}t_1^2 t_2 - t_1 e^{t_2} + \frac{1}{2}t_1^2 \log t_1.$$

Besides,

$$E = t_1 \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \qquad \Longleftrightarrow \qquad E(\lambda) = \lambda - p\lambda_p,$$

$d = 1$ and $e = \frac{1}{t_1 + e^{t_2}}\left(t_1 \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right)$, since $e(\lambda) = 1 + \frac{w}{u-w}\lambda_p$. This is example when the unit field e is not flat. This particular example of Frobenius manifold was considered very recently in [7].

Example 6.24. The case of $s=0$ and A_M consisting of meromorphic function with pole of order one at infinity ($n=1$) and $v\equiv t_1$ ($m_1=1$):

$$\lambda = p + \frac{t_3}{p-t_1} + \frac{t_2^2}{(p-t_1)^2},$$

where the flat coordinates are: $t_1 \equiv \frac{1}{2}t_v^2$, $t_2\equiv \frac{1}{2}t_v^1$ and $t_3\equiv t_v^0$. One obtains the prepotential

$$\mathcal{F} = t_1 t_2^2 + \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_3^2 \log t_2,$$

with

$$E = t_1\frac{\partial}{\partial t_1} + \frac{3}{2}t_2\frac{\partial}{\partial t_2} + 2t_3\frac{\partial}{\partial t_3} \qquad \Longleftrightarrow \qquad E(\lambda) = \lambda - p\lambda_p,$$

$d=-1$ and $e=\frac{\partial}{\partial t_1}$ as $e(\lambda) = 1-\lambda_p$.

Example 6.25. The case of $s=1$, and six-dimensional Frobenius manifold associated with $\mathcal{A}_M$ for $n=m_0=2$ and $m_1=1$. Thus,

$$\lambda = p^2 + t_4\,p + t_1 + t_5 + \frac{t_2\,e^{\frac{t_3}{2}}}{p} + \frac{e^{t_3}}{p^2} - \frac{t_5\,e^{t_6}}{p+e^{t_6}},$$

where the flat coordinates are $t_1\equiv t_0^0$, $t_2\equiv t_0^1$, $t_3\equiv t_0^2$, $t_4\equiv t_\infty^1$, $t_5\equiv t_v^0$ and $t_6\equiv t_v^1$ ($v\equiv -e^{t_6}$). One obtains

$$\begin{aligned}\mathcal{F} = &-\frac{t_2^4}{96} + \frac{1}{4}t_1t_2^2 - \frac{t_4^4}{96} + \frac{1}{4}t_1t_4^2 + \frac{1}{4}t_1^2t_3 + \frac{1}{4}t_4^2t_5 + \frac{1}{2}t_5^2t_6 + t_1t_5t_6 + t_2t_4e^{\frac{t_3}{2}}\\ &+\frac{e^{t_3}}{2} + \frac{1}{2}t_5e^{2t_6} - t_4t_5e^{t_6} + t_2t_5e^{\frac{t_3}{2}-t_6} - \frac{1}{2}t_5e^{t_3-2t_6} + \frac{1}{2}t_5^2\log t_5\end{aligned}$$

and

$$E = t_1\frac{\partial}{\partial t_1} + \frac{1}{2}t_2\frac{\partial}{\partial t_2} + 2\frac{\partial}{\partial t_3} + \frac{1}{2}t_4\frac{\partial}{\partial t_4} + t_5\frac{\partial}{\partial t_5} + \frac{1}{2}\frac{\partial}{\partial t_6} \qquad \Longleftrightarrow \qquad E(\lambda) = \lambda - \frac{1}{2}p\lambda_p,$$

$d=1$ and $e=\frac{\partial}{\partial t_1}$ as $e(\lambda)=1$.

Appendix A. Convention and notation

Here, we fix convention and notation as well as provide useful or required in the main text facts from the rather standard differential geometry, all in a coordinate-free form.

Directional derivative. Let M be a smooth manifold. The directional derivative at point $x\in M$ (in some chart) of a tensor field T is

$$(\mathrm{D}_X T)(x) = \frac{d}{d\varepsilon}T(x+\varepsilon X(x))\Big|_{\varepsilon=0} \qquad x\in M,$$

where $X\in\mathfrak{X}(M)$ is some vector field. Then,

$$\mathrm{D}_X f = X(f) \equiv \langle df, X\rangle, \tag{A.1}$$

where $\langle\cdot,\cdot\rangle : \Lambda^1(M)\times\mathfrak{X}(M)\to\mathcal{C}^\infty(M)$ is the ordinary duality pairing, $f\in\mathcal{C}^\infty(M)$ is a smooth function and $df\in\Lambda^1(M)$ is its differential. The Lie bracket between two vector fields X and Y takes the form

$$[X,Y]=\mathrm{D}_XY-\mathrm{D}_YX. \tag{A.2}$$

Using the above definition one can show that

$$\mathrm{D}_{[X,Y]}T=\mathrm{D}_X\,\mathrm{D}_YT-\mathrm{D}_Y\,\mathrm{D}_XT. \tag{A.3}$$

In fact, the directional derivative is the simplest example of a symmetric and flat covariant derivative.

Lie derivative. Let $X\in\mathfrak{X}(M)$. The Lie derivative of functions coincides with the directional derivative (A.1), $\mathrm{Lie}_X f=\mathrm{D}_Xf$. The Lie derivative of vector fields coincides with the Lie bracket (A.2), $\mathrm{Lie}_X Y=[X,Y]$. Through the Leibniz rule its action can be uniquely extended on any tensor field.

Proposition A.1. *Let $\gamma\in\Lambda^1(M)$. Then,*

$$\mathrm{Lie}_X\gamma=\mathrm{D}_X\gamma+\mathrm{D}^*_\gamma X, \tag{A.4}$$

where $\langle\mathrm{D}^*_\gamma X,Y\rangle:=\langle\gamma,\mathrm{D}_YX\rangle$. The Lie derivative of a tensor field $\circ:\Lambda^1(M)\otimes\Lambda^1(M)\to\Lambda^1(M)$, $\circ(\alpha,\beta)\equiv\alpha\circ\beta$, is

$$(\mathrm{Lie}_X\circ)\,(\alpha,\beta)=(\mathrm{D}_X\circ)\,(\alpha,\beta)+\mathrm{D}^*_{\alpha\circ\beta}X-\mathrm{D}^*_\alpha X\circ\beta-\alpha\circ\mathrm{D}^*_\beta X. \tag{A.5}$$

Proof. From the Leibniz rule we have

$$\langle\mathrm{Lie}_X\gamma,Y\rangle=\mathrm{Lie}_X\langle\gamma,Y\rangle-\langle\gamma,\mathrm{Lie}_XY\rangle=\langle\mathrm{D}_X\gamma,Y\rangle+\langle\gamma,\mathrm{D}_YX\rangle.$$

Hence, (A.4) follows. In a similar fashion, using (A.4) and

$$\begin{aligned}\langle(\mathrm{Lie}_X\circ)(\alpha,\beta),Y\rangle&=\mathrm{Lie}_X\langle\alpha\circ\beta,Y\rangle-\langle\alpha\circ\beta,\mathrm{Lie}_XY\rangle\\&\qquad-\langle\mathrm{Lie}_X\alpha\circ\beta,Y\rangle-\langle\alpha\circ\mathrm{Lie}_X\beta,Y\rangle\end{aligned}$$

one obtains (A.5). □

Levi-Civita connection. Let the manifold M be equipped with a (pseudo-Riemannian) covariant metric $\eta\in\Gamma(S^2T^*M)$. Then, η and its inverse (contravariant metric) $\eta^*\in\Gamma(S^2TM)$ induces canonical isomorphisms

$$\begin{aligned}&\sharp:\Lambda^1(M)\to\mathfrak{X}(M)\qquad\alpha\mapsto\alpha^\sharp:=\sharp(\alpha),\\&\flat:\mathfrak{X}(M)\to\Lambda^1(M)\qquad X\mapsto X^\flat:=\flat(X)\end{aligned}$$

such that $\flat\circ\sharp=\sharp\circ\flat=\mathrm{id}$ and

$$\eta^*(\alpha,\beta)=\langle\alpha,\beta^\sharp\rangle=\eta(\alpha^\sharp,\beta^\sharp).$$

The (unique) Levi-Civita connection is a covariant derivative $\nabla:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ fulfilling the following two requirements:

i) ∇ is symmetric (torsionless),

$$\nabla_XY-\nabla_YX=[X,Y]; \tag{A.6}$$

ii) ∇ preserves the metric, that is $\nabla\eta = 0$. Equivalently

$$X(\eta(Y,Z)) = \eta(\nabla_X Y, Z) + \eta(Y, \nabla_X Z)\,. \tag{A.7}$$

Let $\gamma \in \Lambda^1(M)$, then (A.6) is equivalent to

$$d\gamma(X,Y) = \langle \nabla_X\gamma, Y\rangle - \langle \nabla_Y\gamma, X\rangle.$$

Since $d\gamma(X,Y) = \langle \mathrm{D}_X\gamma, Y\rangle - \langle \mathrm{D}_Y\gamma, X\rangle$, the first condition for ∇ becomes

$$\langle \nabla_X\gamma - \mathrm{D}_X\gamma, Y\rangle = \langle \nabla_Y\gamma - \mathrm{D}_Y\gamma, X\rangle.$$

The second condition can be rewritten in the form

$$(\mathrm{D}_X\eta^*)(\beta,\gamma) = \eta^*(\nabla_X\beta - \mathrm{D}_X\beta, \gamma) + \eta^*(\beta, \nabla_X\gamma - \mathrm{D}_X\gamma)\,,$$

where $\beta, \gamma \in \Lambda^1(M)$.

Using the canonical isomorphism between tangent and cotangent bundles, induced by the metric η, we can define

$$\Gamma_\alpha\gamma \equiv \Gamma(\alpha,\beta) := \mathrm{D}_{\alpha^\sharp}\gamma - \nabla_{\alpha^\sharp}\gamma \qquad \alpha,\gamma \in \Lambda^1(M). \tag{A.8}$$

It is a tensor field since Γ is obviously $\mathcal{C}^\infty(M)$-bilinear map. In the coordinates in which the directional and covariant derivatives are taken Γ coincides withe the Christoffel symbols of the Levi-Civita connection. Hence, the following lemma is valid.

Lemma A.2. *A covariant derivative is the Levi-Civita connection iff the following conditions are satisfied:*

$$\eta^*(\alpha, \Gamma_\beta\gamma) = \eta^*(\beta, \Gamma_\alpha\gamma) \tag{A.9}$$

and

$$\left(\mathrm{D}_{\alpha^\sharp}\eta^*\right)(\beta,\gamma) + \eta^*(\Gamma_\alpha\beta, \gamma) + \eta^*(\beta, \Gamma_\alpha\gamma) = 0. \tag{A.10}$$

To calculate the curvature tensor of the Levi-Civita connection we find

$$\nabla_{\alpha^\sharp}\nabla_{\beta^\sharp}\gamma = \mathrm{D}_{\alpha^\sharp}\,\mathrm{D}_{\beta^\sharp}\gamma - \left(\mathrm{D}_{\alpha^\sharp}\Gamma\right)(\beta,\gamma) - \Gamma_{\mathrm{D}_{\alpha^\sharp}\beta}\gamma - \Gamma_\beta\,\mathrm{D}_{\alpha^\sharp}\gamma - \Gamma_\alpha\,\mathrm{D}_{\beta^\sharp}\gamma + \Gamma_\alpha\Gamma_\beta\gamma$$

and

$$\nabla_{[\alpha^\sharp,\beta^\sharp]}\gamma = \mathrm{D}_{[\alpha^\sharp,\beta^\sharp]}\gamma - \Gamma_{[\alpha^\sharp,\beta^\sharp]^\flat}\gamma.$$

Since the connection is torsionless, we have

$$[\alpha^\sharp, \beta^\sharp]^\flat = \nabla_{\alpha^\sharp}\beta - \nabla_{\beta^\sharp}\alpha = \mathrm{D}_{\alpha^\sharp}\beta - \mathrm{D}_{\beta^\sharp}\alpha - \Gamma_\alpha\beta + \Gamma_\beta\alpha. \tag{A.11}$$

Hence, using the relation (A.3) we derive the curvature tensor in the form

$$\begin{aligned} R(\alpha^\sharp,\beta^\sharp)\gamma &= \nabla_{\alpha^\sharp}\nabla_{\beta^\sharp}\gamma - \nabla_{\beta^\sharp}\nabla_{\alpha^\sharp}\gamma - \nabla_{[\alpha^\sharp,\beta^\sharp]}\gamma \\ &= \left(\mathrm{D}_{\beta^\sharp}\Gamma\right)(\alpha,\gamma) - \left(\mathrm{D}_{\alpha^\sharp}\Gamma\right)(\beta,\gamma) + \Gamma_\alpha\Gamma_\beta\gamma - \Gamma_\beta\Gamma_\alpha\gamma - \Gamma_{\Gamma_\alpha\beta}\gamma + \Gamma_{\Gamma_\beta\alpha}\gamma. \end{aligned} \tag{A.12}$$

Appendix B. Hydrodynamic Poisson bracket

Poisson structures of hydrodynamic type. Let η be a flat (pseudo-Riemannian) covariant metric on a manifold M. Then, on the loop space $\mathcal{L}(M) \equiv \mathcal{C}^\infty(\mathbb{S}^1, M)$, that consists of smooth maps of a circle to M, one can define Poisson bracket of the hydrodynamic type:

$$\{H, F\} = \int_{\mathbb{S}^1} \frac{\partial f}{\partial u^i} \pi^{ij} \frac{\partial h}{\partial u^j}\, dx := \int_{\mathbb{S}^1} \langle df, (\nabla_{u_x} dh)^\sharp \rangle dx, \tag{B.1}$$

where

$$H = \int_{\mathbb{S}^1} h(u(x))\, dx, \qquad F = \int_{\mathbb{S}^1} f(u(x))\, dx$$

are functionals on $\mathcal{L}(M)$, $u : \mathbb{S}^1 \to M$, $x \mapsto u(x) = (u^1(x), \ldots, u^n(x))$ and $u^i(x)$ are local coordinate fields, ∇_{u_x} is the Levi-Civita connection of η taken along the vector field $u_x \equiv \frac{du}{dx}$ tangent to a loop. The related Poisson tensor in this local coordinates has the form

$$\pi^{ij} = \eta^{ij} \frac{d}{dx} - \eta^{ik} \Gamma^j_{kl} u^l_x, \tag{B.2}$$

where $\frac{d}{dx} \equiv \frac{du^k}{dx} \frac{\partial}{\partial u^k}$ is the total derivative with respect to $x \in \mathbb{S}^1$. Important is fact that the invariance with respect to change of coordinates is preserved on the level of the infinite-dimensional formalism of the hydrodynamic Poisson brackets. Recall that (B.1) is the well-known Dubrovin-Novikov bracket [18]. Field bracket defined, through an operator of the form (B.2), by means of a nondegenerate matrix η^{ij} is a Poisson bracket iff η_{ij} can be interpreted as a flat covariant metric and Γ^j_{kl} as the corresponding Levi-Civita connection.

The Poisson brackets of the form (B.1) naturally arise in the study of Hamiltonian structures for $(1 + 1)$-dimensional hydrodynamic (dispersionless) systems. This class of systems is described by quasi-homogeneous first order PDE's. To any flat pencil of contravariant metrics one can associate integrable hydrodynamic hierarchies with bi-Hamiltonian structure defined by means of hydrodynamic Poisson brackets. As it turns out, under some homogeneity assumptions, natural geometric setting for the formalism of hydrodynamic bi-Hamiltonian structures is the theory of Frobenius manifolds [14, 15].

Principle hierarchy. The functions $\mathcal{H}^k_{(n)}$ from (2.12) taken as densities of corresponding functionals on the loop manifold associated to the Frobenius manifold generate hierarchies of Hamiltonian hydrodynamic systems called as a Principal hierarchy [14]. In flat coordinates of the metric η these hierarchies take the form

$$\frac{\partial t^i}{\partial T^{(n)}_k} = \eta^{ij} \frac{\partial^2 \mathcal{H}^k_{(n)}}{\partial x \partial t^j} = \eta^{ij} \frac{\partial^2 \mathcal{H}^k_{(n)}}{\partial t^j \partial t^k} \frac{\partial t^k}{\partial x} \qquad n \geqslant 0, \tag{B.3}$$

where $t^i = t^i(x)$ are dynamical fields depending on the infinite sets of evolution parameters (times) $T^{(n)}_k$. The recurrence formula (2.12) is a counterpart of the bi-Hamiltonian recursion scheme and in fact the hierarchies (B.3) can be written in a quasi bi-Hamiltonian form with respect to hydrodynamic Poisson brackets generated by the metric η and the intersection form g.

Appendix C. Classical r-matrix formalism on Poisson algebras

Let $(\mathfrak{g},[\cdot,\cdot])$ be a Lie algebra. Classical r-matrix [30] is a linear map $r:\mathfrak{g}\to\mathfrak{g}$ such that

$$[a,b]_r := [r(a),b]+[a,r(b)] \qquad a,b\in\mathfrak{g},$$

defines second Lie bracket on $\mathfrak{g}$. Fulfilling of the modified Yang-Baxter equation:

$$[r(a),r(b)]-r([a,b]_r)+\kappa\,[a,b]=0 \qquad \kappa\in\mathbb{K},$$

is a sufficient condition for a linear map r to be a classical r-matrix. Simplest solutions can be obtained through appropriate decomposition of $\mathfrak{g}$ into Lie subalgebras, that is $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-$. Then, $r=\frac{1}{2}(P_+-P_-)$, where P_+,P_- are respective projections onto respective Lie subalgebras, is a classical r-matrix, since it satisfies the Yang-Baxter equation for $\kappa=\frac{1}{4}$.

Theorem C.1 ([26])**.** *Let $(\mathfrak{a},\{\cdot,\cdot\},\cdot)$ have a structure of Poisson algebra, that is $\mathfrak{a}$ is unital, commutative algebra and the Lie bracket $\{\cdot,\cdot\}$ is a derivation with respect to the multiplication. Assume also existence of a non-degenerate* ad*-invariant scalar product $(a,b)_{\mathfrak{a}}\equiv \mathrm{Tr}(ab)$ on $\mathfrak{a}$, which means that $\mathfrak{a}^*\cong\mathfrak{a}$ and $\mathrm{ad}^*\cong\mathrm{ad}$. Then, if $r:\mathfrak{a}\to\mathfrak{a}$ is a classical r-matrix, the formula*

$$\{h,f\}_n(\lambda)=(\lambda,\{r(\lambda^n dh),df\}+\{dh,r(\lambda^n df)\})_{\mathfrak{a}} \qquad h,f\in\mathcal{F}(\mathfrak{a}),$$

defines for each $n\geqslant 0$ Poisson bracket. Moreover, all these brackets are mutually compatible.

The related Poisson tensors π_n, such that $\{h,f\}_n=(df,\pi_n dh)_{\mathfrak{a}}$, have the following form

$$\pi_n dh=\{\lambda,r(\lambda^n dh)\}+\lambda^n r^*(\{\lambda,dh\})\,, \tag{C.1}$$

where the adjoint of r is defined by $(r^*(a),b)_{\mathfrak{a}}:=(a,r(b))_{\mathfrak{a}}$.

Appendix D. Integrals involving logarithmic singularities

Here, we explain how to deal with the residuum integrals from (6.20) and (6.21) involving logarithmic singularities, that is integrals with branch points [1]. We want to compute the following integral for $\mu\geqslant 0$:

$$I:=\mathrm{Tr}_{v_k}(\lambda^\mu\log\lambda)+\frac{m_k}{n}\,\mathrm{Tr}_\infty(\lambda^\mu\log\lambda) \qquad s=0,1,$$

where λ is meromorphic function with poles at ∞ and v_i for $1-s\leqslant i\leqslant K$ ($v_0\equiv 0$ if $s=1$). Recall that Tr is given by (6.2). In the first integral we write $\log\lambda=\log[(p-v_k)^{m_k}\lambda]-m_k\log(p-v_k)$ and in the second integral: $\log\lambda=\log[(p-v_k)^{-n}\lambda]+n\log(p-v_k)$. Hence,

$$I=\mathrm{Tr}_{v_k}(\lambda^\mu\log[(p-v_k)^{m_k}\lambda])+\frac{m_k}{n}\,\mathrm{Tr}_\infty\big(\lambda^\mu\log\big[(p-v_k)^{-n}\lambda\big]\big)+m_k\,I_0,$$

where

$$I_0=-\,\mathrm{Tr}_{v_k}(\lambda^\mu\log(p-v_k))+\mathrm{Tr}_\infty(\lambda^\mu\log(p-v_k))\,.$$

One can now write these two integrals in I_0 as one with contour surrounding a branch cut between v_k and ∞. Now, after applying the residue theorem we have

$$I_0 = \sum_{\substack{1-s\leqslant j\leqslant K\\ j\neq k}} \mathrm{Tr}_{v_j}\left(\lambda^\mu \log(p - v_k)\right).$$

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